CHAPTER 9

Exponential Functions

Section 9A

Exponential Functions

- 9-1 Geometric Sequences
- 9-2 Exponential Functions
- Connecting Algebra to Geometry Changing Dimensions

Section 9B

Functions, Models, and Patterns

- Algebra Lab Model Growth and Decay
- 9-3 Exponential Growth and Decay
- Extension Patterns and Recursion
- 9-4 Linear, Quadratic, and Exponential Models
- Extension Linear and Nonlinear Rates of Change
- 9-5 Comparing Functions

Pacing Guide for 45-Minute Classes

Teacher One Stop™ Calendar Planner®

Chapter 9

DAY 1	DAY 2	DAY 3	DAY 4	DAY 5
9-1 Lesson	9-2 Lesson	Connecting Algebra to Geometry	Algebra Lab	9-3 Lesson
DAY 6	**DAY 7**	**DAY 8**	**DAY 9**	**DAY 10**
9-3 Lesson	Extension	9-4 Lesson	9-4 Lesson	Extension
DAY 11	**DAY 12**	**DAY 13**	**DAY 14**	**DAY 15**
9-5 Lesson	9-5 Lesson	Multi-Step Test Prep Ready to Go On?	Chapter Review	Chapter Test

Pacing Guide for 90-Minute Classes

Teacher One Stop™ Calendar Planner®

Chapter 9

DAY 1	DAY 2	DAY 3	DAY 4	DAY 5
9-1 Lesson 9-2 Lesson	Connecting Algebra to Geometry Algebra Lab	9-3 Lesson	Extension 9-4 Lesson	9-4 Lesson Extension 9-5 Lesson
DAY 6	**DAY 7**			
9-5 Lesson Multi-Step Test Prep Ready to Go On?	Chapter Review Chapter Test			

ONGOING ASSESSMENT and INTERVENTION

	DIAGNOSE	PRESCRIBE
Assess Prior Knowledge	**Before the Chapter**	
	Diagnose readiness for the chapter.	Prescribe intervention.
	Are You Ready? SE	***Are You Ready? Intervention***
Formative Assessment	**Before Every Lesson**	
	Diagnose readiness for the lesson.	Prescribe intervention.
	Warm Up TE	**Reteach** CRB
	During Every Lesson	
	Diagnose understanding of lesson concepts.	Prescribe intervention.
	Check It Out! SE	**Reading Strategies** CRB
	Questioning Strategies TE	***Success for Every Learner***
	Think and Discuss SE	***Lesson Tutorial Videos***
	Write About It SE	
	Journal TE	
	After Every Lesson	
	Diagnose mastery of lesson concepts.	Prescribe intervention.
	Lesson Quiz TE	**Reteach** CRB
	Test Prep SE	**Test Prep Doctor** TE
	ExamView™ Assessment Suite	***Homework Help* Online**
	Before Chapter Testing	
	Diagnose mastery of concepts in chapter.	Prescribe intervention.
	Ready to Go On? SE	***Ready to Go On? Intervention***
	Multi-Step Test Prep SE	**Scaffolding Questions** TE
	Section Quizzes AR	**Reteach** CRB
	ExamView™ Assessment Suite	***Lesson Tutorial Videos***
	Before High Stakes Testing	
	Diagnose mastery of benchmark concepts.	Prescribe intervention.
	College Entrance Exam Practice SE	***College Entrance Exam Practice***
	Standardized Test Prep SE	
Summative Assessment	**After the Chapter**	
	Check mastery of chapter concepts.	Prescribe intervention.
	Multiple-Choice Tests (Forms A, B, C)	**Reteach** CRB
	Free-Response Tests (Forms A, B, C)	***Lesson Tutorial Videos***
	Performance Assessment AR	
	Cumulative Test AR	
	ExamView™ Assessment Suite	

KEY: **SE** = *Student Edition* **TE** = *Teacher's Edition* **CRB** = *Chapter Resource Book* **AR** = *Assessment Resources* Available online

CHAPTER 9

Lesson Resources

Before the Lesson

Prepare

Teacher One Stop
- Editable lesson plans
- Calendar Planner
- Easy access to all chapter resources

Lesson Transparencies
- Teacher Tools

Teach the Lesson

Introduce

Alternate Openers: Explorations

Lesson Transparencies
- Warm Up
- Problem of the Day

Teach

Lesson Transparencies
- Teaching Transparencies

Know-It Notebook™
- Vocabulary
- Key Concepts

Power Presentations

Lesson Tutorial Videos SPANISH

Interactive Online Edition
- Lesson Activities
- Lesson Tutorial Videos

Lab Activities

Lab Resources Online

Online Interactivities SPANISH

TechKeys

Practice the Lesson

Practice

Chapter Resources
- Practice A, B, C

Practice and Problem Solving Workbook

IDEA Works!® Modified Worksheets and Tests

ExamView™ Assessment Suite

Homework Help Online

Online Interactivities SPANISH

Interactive Online Edition
- Homework Help

Apply

Chapter Resources
- Problem Solving

Practice and Problem Solving Workbook

Interactive Online Edition
- Chapter Project

Project Teacher Support

After the Lesson

Reteach

Chapter Resources
- Reteach
- Reading Strategies ELL

Success for Every Learner

Review

Interactive Answers and Solutions

Solutions Key

Know-It Notebook™
- Big Ideas
- Chapter Review

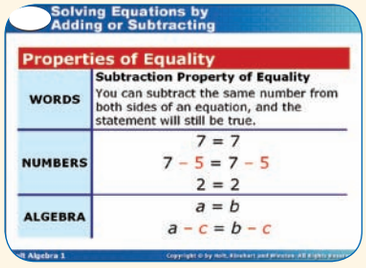
Extend

Chapter Resources
- Challenge

Technology Highlights for the Teacher

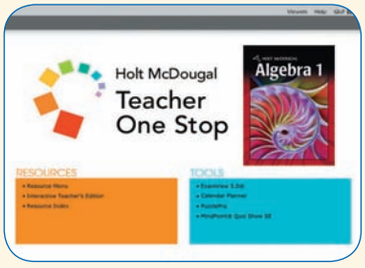
Power Presentations

Dynamic presentations to engage students. Complete PowerPoint® presentations for every lesson in this chapter.



Teacher One Stop SPANISH

Easy access to the chapter resources and assessments. Includes lesson planning software.



Interactive Online Edition

This chapter includes Tutorial Videos, Lesson Activities, Lesson Quizzes, Homework Help, and Chapter Project.

KEY: **SE** = *Student Edition* **TE** = *Teacher's Edition* ELL English Language Learners SPANISH Spanish version available Available online

CHAPTER 9

Reaching All Learners

Teaching tips to help all learners appear throughout the chapter. A few that target specific students are included in the lists below.

All Learners

Lab Activities
Practice and Problem Solving Workbook
Know-It Notebook

Special Needs Students

Practice A ..... CRB
Reteach ..... CRB
Reading Strategies ..... CRB
Are You Ready? ..... SE
Inclusion ..... TE
IDEA Works!® Modified Worksheets and Tests
Ready to Go On? Intervention
Know-It Notebook
Online Interactivities SPANISH
Lesson Tutorial Videos SPANISH

Developing Learners

Practice A ..... CRB
Reteach ..... CRB
Reading Strategies ..... CRB
Are You Ready? ..... SE
Vocabulary Connections ..... SE
Questioning Strategies ..... TE
Ready to Go On? Intervention
Know-It Notebook
Homework Help Online
Online Interactivities SPANISH
Lesson Tutorial Videos SPANISH

On-Level Learners

Practice B ..... CRB
Problem Solving ..... CRB
Vocabulary Connections ..... SE
Questioning Strategies ..... TE
Ready to Go On? Intervention
Know-It Notebook
Homework Help Online
Online Interactivities SPANISH

Advanced Learners

Practice C ..... CRB
Challenge ..... CRB
Challenge Exercises ..... SE
Reading and Writing Math *Extend* ..... TE
Are You Ready? Enrichment
Ready To Go On? Enrichment

English Language Learners

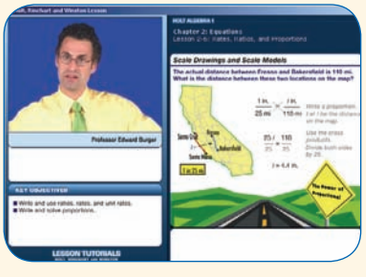
ENGLISH LANGUAGE LEARNERS

Reading Strategies ..... CRB
Are You Ready? Vocabulary ..... SE
Vocabulary Connections ..... SE
Vocabulary Review ..... SE
English Language Learners ..... TE
Success for Every Learner
Know-It Notebook
Multilingual Glossary
Lesson Tutorial Videos SPANISH

Technology Highlights for Reaching All Learners

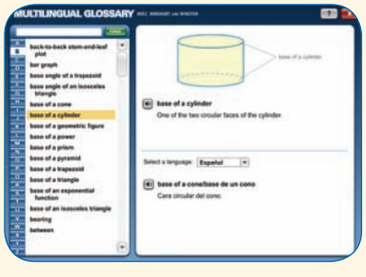
Lesson Tutorial Videos SPANISH

Starring Holt authors Ed Burger and Freddie Renfro! Live tutorials to support every lesson in this chapter.



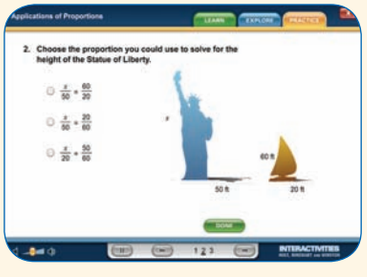
Multilingual Glossary

Searchable glossary includes definitions in English, Spanish, Vietnamese, Chinese, Hmong, Korean, and other languages.



Online Interactivities

Interactive tutorials provide visually engaging alternative opportunities to learn concepts and master skills.



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CHAPTER 9

Ongoing Assessment

Assessing Prior Knowledge

Determine whether students have the prerequisite concepts and skills for success in the chapter.

Are You Ready? ..... SE
Warm Up ..... TE

Test Preparation

Provide review and practice for the chapter and standardized tests.

Multi-Step Test Prep ..... SE
Study Guide: Review ..... SE
Test Tackler ..... SE
Standardized Test Prep ..... SE
College Entrance Exam Practice ..... SE
Countdown to Mastery ..... SE
IDEA Works!® Modified Worksheets and Tests

Alternative Assessment

Assess students' understanding of the chapter concepts and combined problem-solving skills.

Chapter Project ..... SE
Alternative Assessment ..... TE
Performance Assessment ..... AR
Portfolio Assessment ..... AR

Lesson Assessment

Provide formative assessment for each lesson of the chapter.

Questioning Strategies ..... TE
Think and Discuss ..... SE
Check It Out! Exercises ..... SE
Write About It ..... SE
Journal ..... TE
Lesson Quiz ..... TE
Alternative Assessment ..... TE
IDEA Works!® Modified Worksheets and Tests

Weekly Assessment

Provide formative assessment for each section of the chapter.

Multi-Step Test Prep ..... SE
Ready to Go On? ..... SE
Section Quizzes ..... AR
ExamView™ Assessment Suite

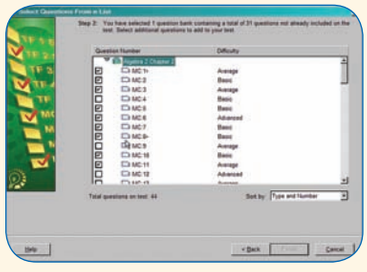
Chapter Assessment

Provide summative assessment of the chapter mastery.

Chapter Test ..... SE
Chapter Test (Levels A, B, C) ..... AR
• Multiple Choice • Free Response
Cumulative Test ..... AR
ExamView™ Assessment Suite
IDEA Works!® Modified Worksheets and Tests

Technology Highlights for Assessment

ExamView™ Assessment Suite
Use the chapter problem banks to create assessments and worksheets to print out or deliver online. Includes dynamic problems.



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CHAPTER 9

Formal Assessment

Three levels (A, B, C) of multiple-choice and free-response chapter tests, along with a performance assessment, are available in the *Assessment Resources.*

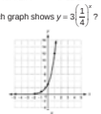
A Chapter Test

C Chapter Test

MULTIPLE CHOICE

B Chapter Test

Select the best answer.

1. Find the next three terms in this sequence: 5120, 1280, 320, 80, …
   A −160, −400, −640
   B 20, 5, 1.25
   C 40, 20, 10
   D 76, 72, 68
2. The first term of a geometric sequence is −2 and the common ratio is 3. What is the 12th term of the sequence?
   F −1,062,882 H −6144
   G −354,294 J 12,288
3. Which graph shows $y = 3\left(\frac{1}{4}\right)^x$?
   A
   
   B

   C

   D

4. The function $f(x) = 50(1.2)^x$ gives the number of bacteria in a science experiment, where x is the number of days after the start of the experiment. To the nearest whole number, how many bacteria will there be after 5 days?
   F 124 H 375,000,000
   G 300 J 777,600,000
5. Which are the next three terms in the geometric sequence 216, 36, 6, 1, …?
   A 0, −1, −6 C $\frac{1}{3}, \frac{1}{6}, \frac{1}{9}$
   B $\frac{1}{6}, \frac{1}{36}, \frac{1}{216}$ D 6, 36, 216
6. Which is the 10th term of the geometric sequence $\frac{1}{512}, \frac{1}{256}, \frac{1}{128}, \frac{1}{64}, \ldots$?
   F 1 H $\frac{1}{4}$
   G $\frac{1}{2}$ J $\frac{1}{8}$
7. Which is the 6th term of the geometric sequence 729, 81, 9,…?
   A $\frac{1}{729}$ C $\frac{1}{9}$
   B $\frac{1}{81}$ D 1
8. Which ordered pairs satisfy an exponential function?
   F

X	−4	−3	−2	−1
Y	0	5	0	−25

   G

X	4	3	2	1
Y	81	27	9	1

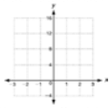
   H

X	4	3	2	1
Y	64	27	8	1

   J

X	4	3	2	1
Y	4	2	0	−2

B Chapter Test *(continued)*

9. Which set of ordered pairs satisfies an exponential function?
   A {(−2, 1), (−8, 2), (−32, 3), (−128, 4)}
   B {(−1, 2), (0, 10), (1, 50), (2, 250)}
   C {(1, 2), (1, 6), (1, 18), (1, 54)}
   D {(1, 2), (2, 4), (3, 6), (4, 8)}
10. The number of members in a labor union is 240, and the number increases by 5% each year. Find the number of members after 4 years.
    F 278 H 810
    G 292 J 1215
11. Iodine-131 has a half-life of approximately 8 days. Find the amount of iodine-131 left from a 100-gram sample after 16 days.
    A 0.0015 g C 25 g
    B 0.39 g D 32.75 g
12. Determine which kind of model best describes this data set.
    {(−2, 1), (−1, 2), (0, 4), (1, 8), (2, 16)}
    
    F linear H quadratic
    G exponential J none
13. Which of the following data sets is best described by a linear model?
    A {(−2, 4), (−1, 9), (0, 16), (1, 25)}
    B {(−2, −1), (−1, 0), (0, 1), (1, 0)}
    C {(−2, 12), (−1, 10), (0, 8), (1, 6)}
    D {(−2, 1), (−1, 0), (0, 1), (1, 4)}
14. Which of the following models best describes the data set?
    {(5, 2), (6, 0), (7, 2), (8, 6)}
    F linear H exponential
    G quadratic J none
15. Martin has $200 in his savings account. He wants to save more money. He is looking at two investment plans. Under plan A, he will increase his account balance by $40 a year. Under plan B, he will increase his account balance by 15% each year. How much more will he save with Plan B after 10 years?
    A $209 C $26
    B $92 D $2
16. Which is the average rate of change over the interval [2, 3]?
    Equation A

X	0	1	2	3
Y	1	4	9	16

    Equation B
    $f(x) = -x + 2$
    F A: 7, B: 1 H A: 3, B: −1
    G A: 7, B: −1 J A: −1, B: 7
17. Which is the y-intercept of the equations?
    Equation A

X	0	1	2	3
Y	1	4	9	16

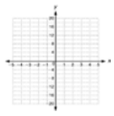
    Equation B
    $f(x) = -x + 2$
    A A: −1, B: −1 C A: −1, B: 2
    B A: 2, B: 0 D A: 1, B: 2

FREE RESPONSE

B Chapter Test

1. Find the next three terms in this sequence: 224, 112, 56, 28, …
2. The first term of a geometric sequence is 5 and the common ratio is −2. What is the 10th term of the sequence?
3. Graph $y = -6(3)^x$.

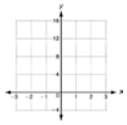
x	y



4. The function $f(x) = 30(1.4)^x$ gives the number of bacteria in a science experiment, where x is the number of days after the start of the experiment. To the nearest whole number, how many bacteria will there be after 4 days?
5. Which are the next three terms in the geometric sequence 4, 2, 1, 0.5, …?
6. Which is the 10th term of the geometric sequence 256, 128, 64, 16 , …?
7. Which is the 6th term of the geometric sequence $\frac{1}{3}$, 1, 3,…?
8. Fill in the ordered pairs that satisfy the exponential function.

X	Y
1	$\frac{1}{64}$
2	
3	$\frac{1}{4}$
4	
5	4
6	16

B Chapter Test *(continued)*

9. Does this set of ordered pairs satisfy an exponential function? Explain.
   {(−2, 0.25), (−1, 0.5), (0, 1.5), (1, 6)}
10. The number of students at a university is 1320, and the number increases by 8% each year. Write an exponential function to model this situation. Then find about how many students there will be in 5 years.
11. Bismuth-212 has a half-life of 60.5 minutes. Find the amount of bismuth-212 left from a 100-gram sample after 242 minutes.
12. Determine which kind of model—linear, quadratic, exponential, or square-root—best describes this data set.
    {(−2, −2), (−1, −1), (0, 2), (1, 7), (2, 14)}
    
13. Which kind of model best describes the graph?

14. Which kind of model best describes the data set?

X	1	2	3	4
Y	0.5	0.25	0	−0.25

15. Emil has $500 in his savings account. He wants to save more money. He is looking at two investment plans. Under plan A, he will increase his account balance by $150 a year. Under plan B, he will increase his account balance by 25% each year. How much more will he save with Plan B after four years? Round your answer to the nearest whole.
16. What is the average rate of change over the interval [−1, 1]?
    Equation A

X	−1	0	1	2
Y	6	4	6	12

    Equation B
    $f(x) = 2x - 3$

PERFORMANCE ASSESSMENT

Chapter Test

Square Sequence
Purpose
This performance task assesses the student's ability to find the *n*th term of a geometric sequence.

Time
20–40 minutes

Grouping
Individuals

Preparation Hints
Point out that the side lengths of each square is twice the length of the one before it. If necessary, review the perimeter and area formulas for a square.

Overview
The student first uses the formula for the perimeter of a square to find the perimeters of a sequence of squares; the student writes a rule for the resulting geometric sequence. Then the student uses the area formula to find the areas of each square; the student writes another rule for this geometric sequence. The student uses the rules to find the perimeters and areas of the next squares, and draws them to confirm the results.

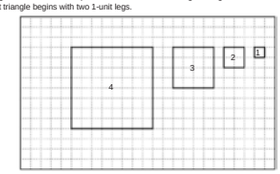
Introduce the Task
As a class, review the formulas for the perimeter and area of a square.

Performance Indicators
_____ Uses the formulas for perimeter and area of a square.
_____ Identifies the first term and common ratio of geometric sequences and writes the formula for the *n*th term.
_____ Uses the area formula of a square.

Scoring Rubric
Level 4: Student generates sequences from the geometric diagram, and extends the sequences with both rules and the diagram.
Level 3: Student generates sequences from the geometric diagram and can extend the diagram, but struggles to write rules.
Level 2: Student can extend the geometric diagram, but cannot generate sequences or rules from it.
Level 1: Student is not able to solve any of the problems.

Chapter Test *(continued)*

Square Sequence
The diagram below shows a spiral created from isosceles right triangles. The first triangle begins with two 1-unit legs.



1. Count grid units and find the perimeter of each square.
   _____ Square 1 _____ Square 2 _____ Square 3 _____ Square 4
2. The perimeters in problem 1 form a geometric sequence. Write the formula for the perimeter of the *n*th square. _____
3. Find the areas of the first four squares.
   _____ Square 1 _____ Square 2 _____ Square 3 _____ Square 4
4. The areas in problem 3 form a geometric sequence. Write the formula for the area of the *n*th square. _____
5. Use your formulas to find the perimeters and areas of the next three squares.
   Square 5 Square 6 Square 7
   perimeters: _____ _____ _____
   areas: _____ _____ _____
6. Draw the next three squares, and calculate the perimeters and areas. Does your drawing confirm your answers to problem 5?
   _____

CHAPTER 9

MATHEMATICAL PRACTICES **Reason abstractly and quantitatively.**

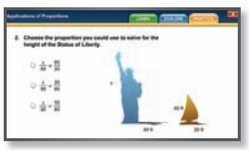
SECTION 9A
Exponential Functions

MULTI-STEP TEST PREP Students write and graph functions to model real-world college tuition situations.

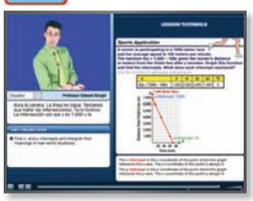
SECTION 9B
Functions, Models, and Patterns

MULTI-STEP TEST PREP Students evaluate exponential functions, and compare linear, quadratic, and exponential models for real-world situations such as population growth.

Interactivities Online ▶



Video Lesson Tutorials Online



Lesson Tutorial Videos are available for EVERY example.

CHAPTER 9

Exponential Functions

9A Exponential Functions

9-1	Geometric Sequences	F.IF.3
9-2	Exponential Functions	F.IF.7e

9B Functions, Models, and Patterns

Lab	Model Growth and Decay	F.LE.2*
9-3	Exponential Growth and Decay	F.LE.2*
Ext	Patterns and Recursion	F.IF.3
9-4	Linear, Quadratic, and Exponential Models	F.LE.1*
Ext	Linear and Nonlinear Rates of Change	F.IF.6*
9-5	Comparing Functions	F.IF.9

COMMON CORE

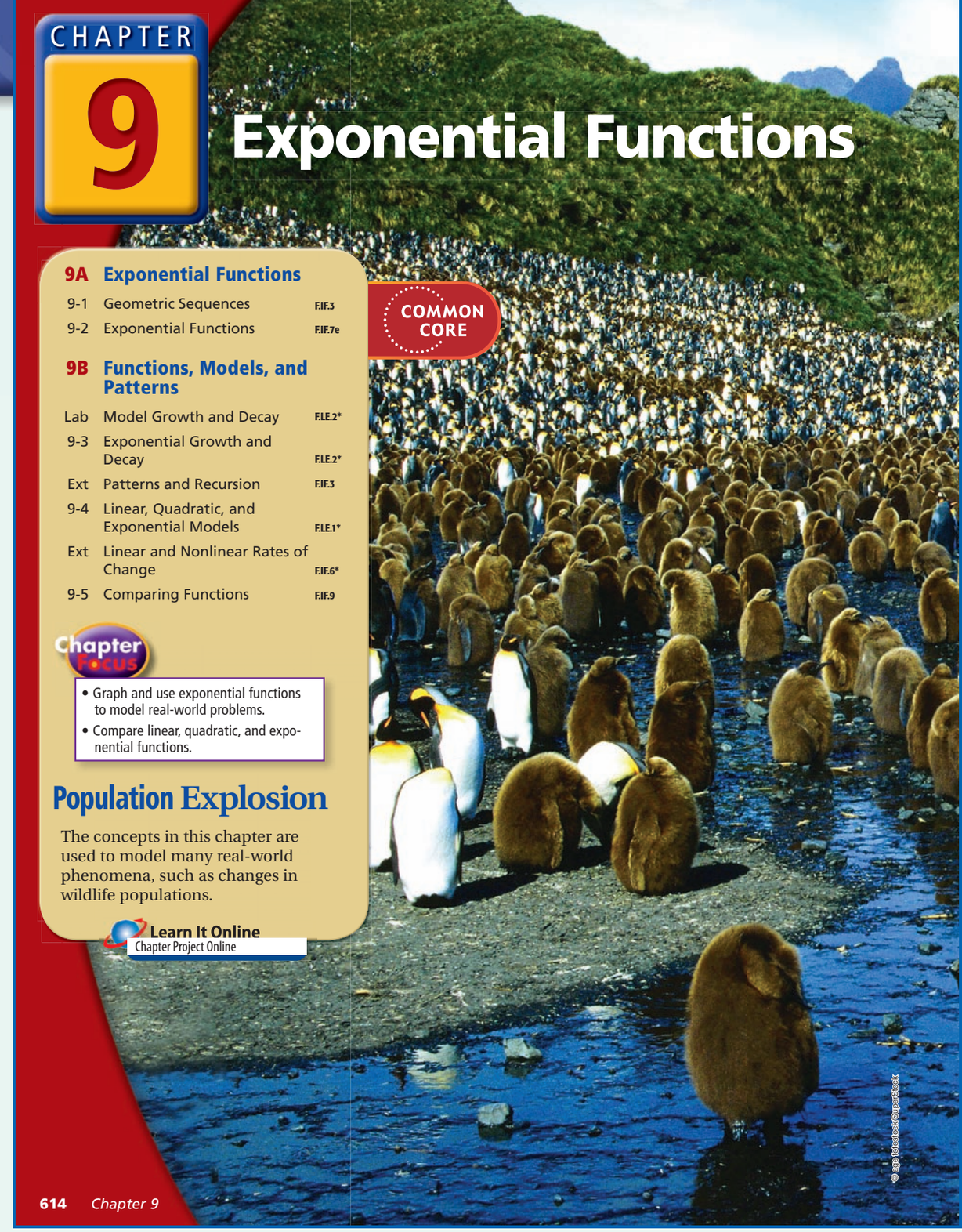
Chapter Focus

- Graph and use exponential functions to model real-world problems.
- Compare linear, quadratic, and exponential functions.

Population Explosion

The concepts in this chapter are used to model many real-world phenomena, such as changes in wildlife populations.

Learn It Online
Chapter Project Online



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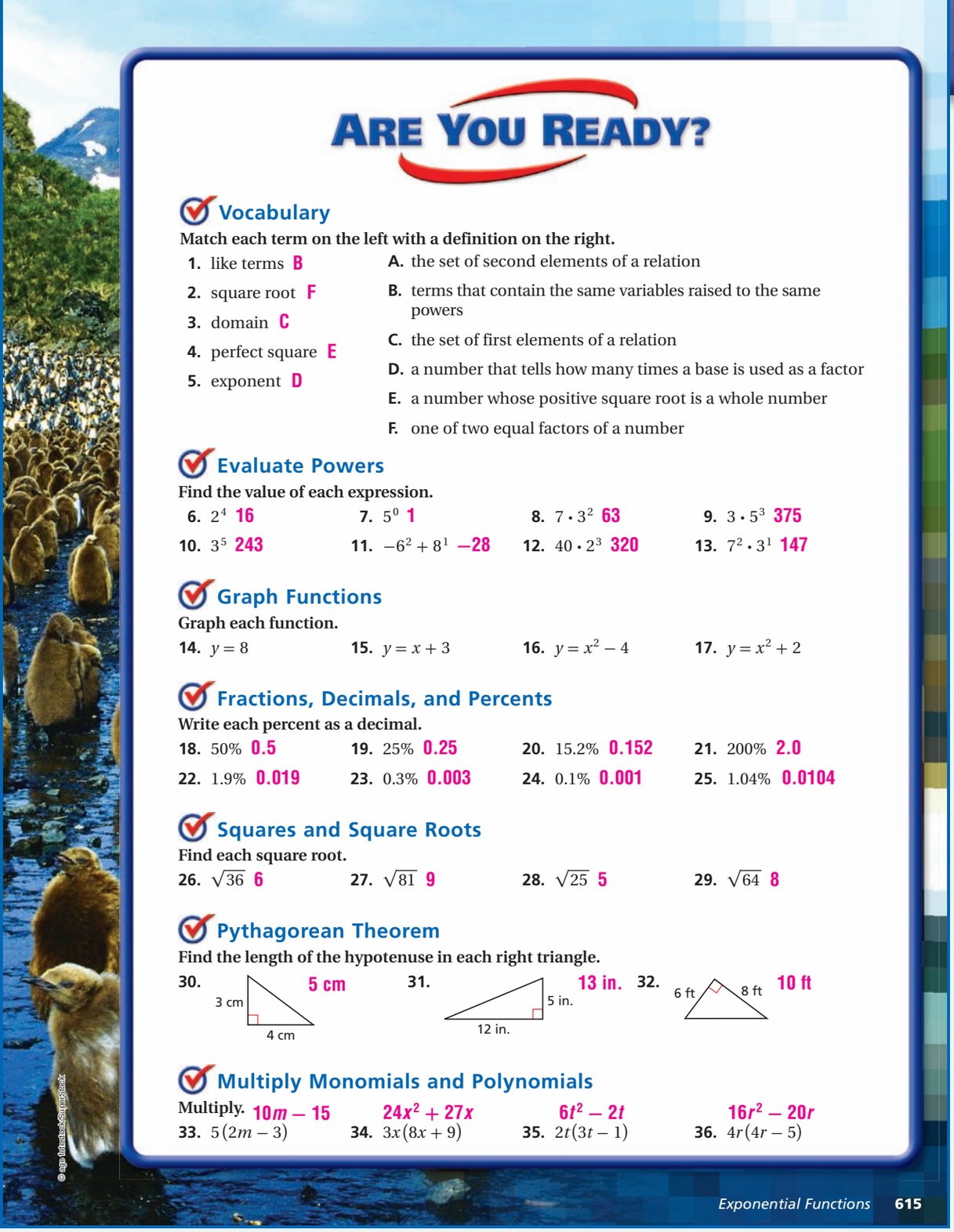
Chapter Project

A project for this chapter is available online.

Project Resources

All project resources for teachers and students are provided online.

Learn It Online
Chapter Project Online



ARE YOU READY?

Vocabulary

Match each term on the left with a definition on the right.

1. like terms **B**
2. square root **F**
3. domain **C**
4. perfect square **E**
5. exponent **D**

A. the set of second elements of a relation

B. terms that contain the same variables raised to the same powers

C. the set of first elements of a relation

D. a number that tells how many times a base is used as a factor

E. a number whose positive square root is a whole number

F. one of two equal factors of a number

Evaluate Powers

Find the value of each expression.

6. 2^4 **16**
7. 5^0 **1**
8. $7 \cdot 3^2$ **63**
9. $3 \cdot 5^3$ **375**
10. 3^5 **243**
11. $-6^2 + 8^1$ **−28**
12. $40 \cdot 2^3$ **320**
13. $7^2 \cdot 3^1$ **147**

Graph Functions

Graph each function.

14. $y = 8$
15. $y = x + 3$
16. $y = x^2 - 4$
17. $y = x^2 + 2$

Fractions, Decimals, and Percents

Write each percent as a decimal.

18. 50% **0.5**
19. 25% **0.25**
20. 15.2% **0.152**
21. 200% **2.0**
22. 1.9% **0.019**
23. 0.3% **0.003**
24. 0.1% **0.001**
25. 1.04% **0.0104**

Squares and Square Roots

Find each square root.

26. $\sqrt{36}$ **6**
27. $\sqrt{81}$ **9**
28. $\sqrt{25}$ **5**
29. $\sqrt{64}$ **8**

Pythagorean Theorem

Find the length of the hypotenuse in each right triangle.

30. **5 cm**

31. **13 in.**

32. **10 ft**

Multiply Monomials and Polynomials

Multiply.

33. $5(2m - 3)$ **$10m - 15$**
34. $3x(8x + 9)$ **$24x^2 + 27x$**
35. $2t(3t - 1)$ **$6t^2 - 2t$**
36. $4r(4r - 5)$ **$16r^2 - 20r$**

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CHAPTER 9

ARE YOU READY?

Organizer

Objective: Assess students' understanding of prerequisite skills.

Assessing Prior Knowledge

INTERVENTION

Diagnose and Prescribe

Use this page to determine whether intervention is necessary or whether enrichment is appropriate.

Resources

***Are You Ready? Intervention and Enrichment* Worksheets**

***Are You Ready?* Online**

Answers

14–17. See Additional Answers.

ARE YOU READY?

Diagnose and Prescribe

NO INTERVENE

YES ENRICH

	***Are You Ready? Intervention*, Chapter 9**		
Prerequisite Skill	**Worksheets**	**Online**	
Evaluate Powers	Skill 8	Activity 8	Diagnose and Prescribe Online
Graph Functions	Skill 80	Activity 80	
Fractions, Decimals, and Percents	Skill 14	Activity 14	
Squares and Square Roots	Skill 6	Activity 6	
Pythagorean Theorem	Skill 31	Activity 31	
Multiply Monomials and Polynomials	Skill 62	Activity 62	

***Are You Ready? Enrichment*, Chapter 9**
Worksheets
Online

Organizer

Objective: Help students organize the new concepts they will learn in this chapter.

Online Edition
Multilingual Glossary

Resources

Learn It Online
Multilingual Glossary

Answers to *Vocabulary Connections*

1. There is something about them that is the same; a comparison of two numbers by division; when you divide the terms, the resulting values will be the same.
2. It is the number of times a factor (the base) is multiplied by itself in a power. It might involve a power that has a variable as the exponent.
3. Exponential growth functions get larger as the exponent increases because the factor that is repeatedly multiplied (the base) is larger than 1. Exponential decay functions get smaller as the exponent increases because the factor that is repeatedly multiplied (the base) is less than 1.
4. Simple interest calculates interest as a simple percent of the amount invested. Compound interest repeatedly applies interest to both the initial amount invested and previously-earned interest.

CHAPTER 9 Study Guide: Preview

Where You've Been

Previously, you

- identified and extended arithmetic sequences.
- identified and graphed linear functions and quadratic functions.
- solved linear and quadratic equations.

In This Chapter

You will study

- another type of sequence—geometric sequences.
- compare linear, quadratic, and linear functions
- compare several types of exponential functions

Where You're Going

You can use the skills in this chapter

- to analyze more complicated functions in later math courses, such as Calculus.
- to explore exponential growth and decay models that are used in science.
- to make informed decisions about finances.

Key Vocabulary/Vocabulario

common ratio	razón común
compound interest	interés compuesto
exponential decay	decrecimiento exponencial
exponential function	función exponencial
exponential growth	crecimiento exponencial
geometric sequence	sucesión geométrica

Vocabulary Connections

To become familiar with some of the vocabulary terms in the chapter, consider the following. You may refer to the chapter, the glossary, or a dictionary if you like.

1. What does it mean when several items have something "in common"? What is a ratio? What do you think **common ratio** means?
2. What is an *exponent*? What do you think an **exponential function** might involve?
3. Two types of functions that you will study in this chapter are **exponential growth** and **exponential decay**. What is the difference in the way these two types of functions behave?
4. You have calculated *simple interest*. What is the difference between *simple interest* and **compound interest**?

616 Chapter 9

MATHEMATICAL PRACTICES

The Common Core Standards for Mathematical Practice describe varieties of expertise that mathematics educators at all levels should seek to develop in their students. Opportunities to develop these practices are integrated throughout this program. Some examples are provided below.

1. **Make sense of problems and persevere in solving them.** Pages 621–622, 628–629, 639–642, 651, 653–655, 664–667, 668
2. **Reason abstractly and quantitatively.** Pages 620, 638
3. **Construct viable arguments and critique the reasoning of others.** Pages 620, 622, 627, 629–630, 638, 640–641, 652, 654, 664, 666
4. **Model with mathematics.** Pages 620, 627, 638, 662–663
5. **Use appropriate tools strategically.** Pages 627, 629, 641
6. **Attend to precision.** Pages 641, 667
7. **Look for and make use of structure.** Pages 627, 630
8. **Look for and express regularity in repeated reasoning.** Pages 621–623, 634, 653, 618–620, 644–645, 656–658

Reading and Writing Math

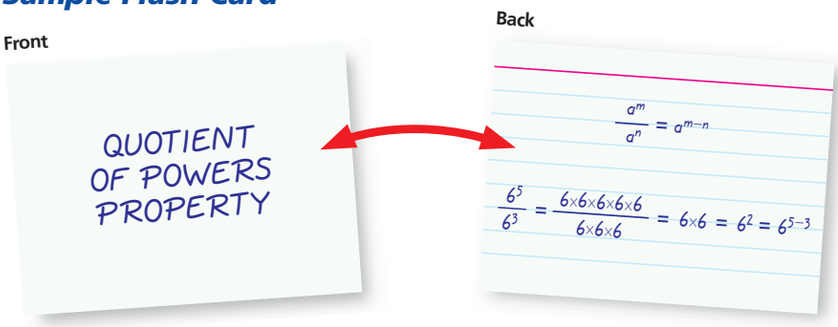
CHAPTER 9

Study Strategy: Remember Properties

In math, there are many formulas, properties, and rules that you should commit to memory.

To memorize a property, create flash cards. Write the name of the property on one side of a card. Write the property on the other side of the card. You might also include a diagram or an example if helpful. Study your flash cards on a regular basis.

Sample Flash Card



Knowing when and how to apply a mathematical property is as important as memorizing the property itself.

To know what property to apply, read the problem carefully and look for key words.

> A professional baseball player earns an annual salary of **$5.7 x 10^6**. There are **162 games** in a baseball season. How much does the player earn **per game**?

The key words have been highlighted. You are asked for an amount *per game* so you need to find a quotient. The annual salary is given in *exponential form* so you will need to use the *quotient of powers property.*

Try This

Read each problem. Then write the property needed to solve it. What key words helped you identify the property?

1. Light travels at about 1.86×10^5 miles per second. Find the approximate distance that light travels in one year if there are about 3.15×10^7 seconds in a year.
2. The area of a rectangular pool is 120 square feet. The length is 1 foot less than twice the width. What is the perimeter of the pool?

COMMON CORE **Standards for Mathematical Content** High School

		LESSON
INTERPRETING FUNCTIONS Understand the concept for a function and use function notation		
CC.9-12.F.IF.3	Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.	9-1, 9-3 Ext
INTERPRETING FUNCTIONS Interpret functions that arise in applications in terms of the context		
CC.9-12.F.IF.6	Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*	9-4 Ext, 9-5
INTERPRETING FUNCTIONS Analyze functions using different representations		
CC.9-12.F.IF.9	Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).	9-5
LINEAR AND EXPONENTIAL MODELS Construct and compare linear and exponential models and solve problems		
CC.9-12.F.LE.1	Distinguish between situations that can be modeled with linear functions and with exponential functions.*	9-2, Lab 9-3, 9-3, 9-4
CC.9-12.F.LE.2	Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (including reading these from a table).*	9-1, Lab 9-3, 9-3, 9-3 Ext

Reading and Writing Math

CHAPTER 9

Organizer

Objective: Help students apply strategies to understand and retain key concepts.

INTERACTIVE **Online Edition**

Study Strategy: Remember Properties

Discuss Suggest that students try simple numerical examples that they can easily check. This may be easier than trying only to use general statements of the properties.

Extend Have students give each other numerical examples of various properties, and then use these to write general property rules.

Answers to *Try This*

Possible answers:

1. Product of powers property; *distance, miles per second,* and *seconds* indicate using the formula $d = rt$, and the rate and time are written using exponents.
2. Distributive property; *area, length,* and *width* indicate using the formula $A = lw$, and "*1 foot less than twice the width*" indicates an expression for length written in parentheses.

Reading Connection

Marvels of Math
by Kendall F. Haven

Haven describes 16 fascinating episodes in the history of mathematics. He introduces each with a short discussion of key terms; tells the stories in short-story form; and follows each story with questions and activities.

Activity After students have studied radical expressions, have them read Chapter 1, *Numbers in the Sand.* Ask them to write about the discovery made by the followers of Pythagoras, and to explain how this discovery contradicted Pythagoras's teachings. Pythagoras believed that all numbers could be expressed as a ratio of whole numbers. His followers discovered $\sqrt{2}$, which cannot be expressed in this way.

SECTION
9A Exponential Functions

One-Minute Section Planner

Lesson	Lab Resources	Materials
Lesson 9-1 Geometric Sequences • Recognize and extend geometric sequences. • Find the *n*th term of a geometric sequence. ☐ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests		**Required** graphing calculator
Lesson 9-2 Exponential Functions • Evaluate exponential functions. • Identify and graph exponential functions. ☐ SAT-10 ☑ NAEP ☐ ACT ☑ SAT ☑ SAT Subject Tests		**Required** graphing calculator

Note: If NAEP is checked, the content is tested on either the Grade 8 or the Grade 12 NAEP assessment.

MK = *Manipulatives Kit*

Selected Answers in Student Edition

The answers to the following exercises are included in the Selected Answers section at the back of the Student Edition textbook.

Lesson 1: 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33a, 33b, 35, 37, 39, 43a, 43b, 43c, 45, 47, 49, 51, 53

Lesson 2: 1, 3, 17, 19, 21, 23, 35, 41, 43a, 43b, 43c, 45, 47, 49, 51

All answers for the Check it Out! exercises in each lesson and for the Chapter Study Guide: Review are included in the Student Edition.

Math Background

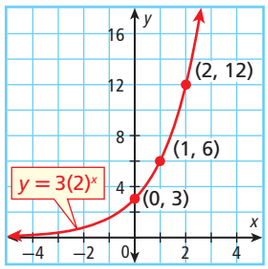
COMMON CORE PROFESSIONAL DEVELOPMENT

GEOMETRIC SEQUENCES AND EXPONENTIAL FUNCTIONS

Lessons 9-1 and 9-2

To understand the connection between geometric sequences and exponential functions, we begin with the definition of *exponential function:* a function of the form $y = ab^x$, where $a \neq 0$, $b > 0$, and $b \neq 1$. At this point, students have seen b^x defined for limited x-values only (namely, rational numbers), but the domain of an exponential function is all real numbers. Students will fill in this gap in future courses and can be assured in the meantime that they may draw a smooth, continuous curve when graphing an exponential function.

For $a > 0$, the range of an exponential function is all positive real numbers, as illustrated in the graph of $y = 3(2)^x$.



When the domain is restricted to the whole numbers, the graph becomes a set of discrete points, the first few of which are shown above. The points' y-values are 3, 6, 12, ... ; this is precisely the *geometric sequence* $a_n = 3(2)^{n-1}$. In other words, a geometric sequence is simply an exponential function with a restricted domain. (Note that the domain is usually restricted to the natural numbers, so that the first term of the sequence corresponds to the input value 1 rather than 0.)

9-1 Organizer

Pacing: Traditional 1 day
Block $\frac{1}{2}$ day

Objectives: Recognize and extend geometric sequences.

Find the *n*th term of a geometric sequence.

Online Edition
Tutorial Videos

Power Presentations with PowerPoint®

Warm Up

Find the value of each expression.

1. 2^5 32
2. 2^{-5} $\frac{1}{32}$
3. -3^4 −81
4. $(-3)^4$ 81
5. $(0.2)^3$ 0.008
6. $7(-4)^2$ 112
7. $15\left(\frac{1}{3}\right)^3$ $\frac{5}{9}$
8. $12(-0.4)^3$ −0.768

Also available online

Math Humor

Q: What type of dress did the rectangle wear to the prom?

A: A dress with geometric sequins.

COMMON CORE

CC.9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.

CC.9-12.F.LE.2 Distinguish between situations that can be modeled with linear functions and with exponential functions.*

c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

CC.9-12.F.BF.2 Write … geometric sequences … with an explicit formula, use them to model situations, ….*

Learn It Online
State Resources Online

COMMON CORE

9-1 Geometric Sequences

CC.9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *Also* **CC.9-12.F.LE.2***, **CC.9-12.F.BF.2***

Objectives
Recognize and extend geometric sequences.

Find the *n*th term of a geometric sequence.

Vocabulary
geometric sequence
common ratio

Who uses this?
Bungee jumpers can use geometric sequences to calculate how high they will bounce.

The table shows the heights of a bungee jumper's bounces.

Bounce	1	2	3
Height (ft)	200	80	32

The height of the bounces shown in the table form a *geometric sequence*. In a **geometric sequence**, the ratio of successive terms is the same number *r*, called the **common ratio**.

Writing Math
The variable *a* is often used to represent terms in a sequence. The variable a_4 (read "*a* sub 4") is the fourth term in a sequence.

Geometric sequences can be thought of as functions. The term number, or position in the sequence, is the input, and the term itself is the output.

Position	1	2	3	4
Term	3	6	12	24
	a_1	a_2	a_3	a_4

To find a term in a geometric sequence, multiply the previous term by *r*.

Finding a Term of a Geometric Sequence

The *n*th term of a geometric sequence with common ratio *r* is

$$a_n = a_{n-1}r$$

EXAMPLE 1 **Extending Geometric Sequences**

Find the next three terms in each geometric sequence.

A 1, 3, 9, 27, …

Step 1 Find the value of *r* by dividing each term by the one before it.

1 → 3 → 9 → 27

$\frac{3}{1} = 3$ $\frac{9}{3} = 3$ $\frac{27}{9} = 3$ ← The value of *r* is 3.

Step 2 Multiply each term by 3 to find the next three terms.

27 → 81 → 243 → 729

× 3 × 3 × 3 $a_n = a_{n-1}r$

The next three terms are 81, 243, and 729.

Mark A. Johnson/photolibrary

1 Introduce

EXPLORATION

Geometric Sequences

Marcella's grandmother wants to begin saving for Marcella's college education. She decides that the first year she will save $1 and that each year thereafter she will save twice as much as the previous year.

1. Complete the table.

Year	1	2	3	4	5	6	7
Amount ($)	1	2	4	8	16	32	64

2. Look at the amounts in the table above. What is the ratio of any amount to the one before it? 2
3. Rewrite the amounts as powers of 2.

Year	1	2	3	4	5	6	7
Amount ($)	2^0	2^1	2^2	2^3	2^4	2^5	2^6

4. How is the exponent in each rewritten amount related to the year? The exponent is 1 less than the year.

THINK AND DISCUSS

5. **Show** how to write a rule that you can use to predict the amount of money saved in year *n*. $A = 2^{n-1}$
6. **Explain** how you can use your rule to find the amount of money saved in year 10. Substitute $n = 10$ in the rule. The amount is $2^{10-1} = 2^9 = 512$.

Motivate

Have students solve the following problem:

You are given 1 cent on the first of 20 days. Each day you will receive twice the amount as the previous day. How much will you receive on the 20th day? $5242.88

Ask students how they figured out the answer. Possible answer: I wrote the amount received for the first day and then doubled the amount for the next 19 days.

Explorations and answers are provided in your online resources.

Helpful Hint

When the terms in a geometric sequence alternate between positive and negative, the value of r is negative.

B $-16, 4, -1, \frac{1}{4}, \ldots$

Step 1 Find the value of r by dividing each term by the one before it.

$-16 \quad 4 \quad -1 \quad \frac{1}{4}$

$\frac{4}{-16} = -\frac{1}{4} \quad \frac{-1}{4} = -\frac{1}{4} \quad \frac{\frac{1}{4}}{-1} = -\frac{1}{4}$ ← The value of r is $-\frac{1}{4}$.

Step 2 Multiply each term by $-\frac{1}{4}$ to find the next three terms.

$\frac{1}{4} \quad -\frac{1}{16} \quad \frac{1}{64} \quad -\frac{1}{256}$

$\times\left(-\frac{1}{4}\right) \quad \times\left(-\frac{1}{4}\right) \quad \times\left(-\frac{1}{4}\right) \quad a_n = a_{n-1}r$

The next three terms are $-\frac{1}{16}$, $\frac{1}{64}$, and $-\frac{1}{256}$.

CHECK IT OUT! **Find the next three terms in each geometric sequence.**

1a. 5, −10, 20, −40, … 80, −160, 320

1b. 512, 384, 288, … 216, 162, 121.5

To find the output a_n of a geometric sequence when n is a large number, you need an equation, or function rule.

The pattern in the table shows that to get the nth term, multiply the first term by the common ratio raised to the power $n - 1$.

Words	Numbers	Algebra
1st term	3	a_1
2nd term	$3 \cdot 2^1 = 6$	$a_1 \cdot r^1$
3rd term	$3 \cdot 2^2 = 12$	$a_1 \cdot r^2$
4th term	$3 \cdot 2^3 = 24$	$a_1 \cdot r^3$
nth term	$3 \cdot 2^{n-1}$	$a_1 \cdot r^{n-1}$

If the first term of a geometric sequence is a_1, the nth term is a_n, and the common ratio is r, then

$$a_n = a_1 r^{n-1}$$

(a_n: nth term; a_1: 1st term; r: Common ratio)

EXAMPLE 2 Finding the nth Term of a Geometric Sequence

A **The first term of a geometric sequence is 128, and the common ratio is 0.5. What is the 10th term of the sequence?**

$a_n = a_1 r^{n-1}$ *Write the formula.*

$a_{10} = 128(0.5)^{10-1}$ *Substitute 128 for a_1, 10 for n, and 0.5 for r.*

$= 128(0.5)^9$ *Simplify the exponent.*

$= 0.25$ *Use a calculator.*

B **For a geometric sequence, $a_1 = 8$ and $r = 3$. Find the 5th term of this sequence.**

$a_n = a_1 r^{n-1}$ *Write the formula.*

$a_5 = 8(3)^{5-1}$ *Substitute 8 for a_1, 5 for n, and 3 for r.*

$= 8(3)^4$ *Simplify the exponent.*

$= 648$ *Use a calculator.*

COMMON ERROR ALERT

When finding a common ratio, students might divide in the wrong order. For instance, in **Example 1B,** students might divide −16 by 4 and so on to get a common ratio of −4. Tell them they are really finding what each term is being *multiplied* by to get the next term, so the inverse operation (division), used in reverse, will produce r.

Power Presentations with PowerPoint®

Additional Examples

Example 1

Find the next three terms in each geometric sequence.

A. 1, 4, 16, 64,… 256, 1024, 4096

B. $-9, 3, -1, \frac{1}{3}, -\frac{1}{9}, \ldots$ $\frac{1}{27}, -\frac{1}{81}, \frac{1}{243}$

Example 2

A. The first term of a geometric sequence is 500 and the common ratio is 0.2. What is the 7th term of the sequence? 0.032

B. For a geometric sequence, $a_1 = 5$ and $r = 2$. Find the 6th term of this sequence. 160

C. What is the 9th term of the sequence 2, −6, 18, −54, …? 13,122

Also available online

2 Teach

Guided Instruction

After reviewing operations with exponents, show students a geometric sequence and how it can be extended with continued multiplication by r. Remind them that an arithmetic sequence is extended with continued addition by d. Choose a sequence from **Example 1** and show how each term is the product of the first term and the common ratio raised to an increasing power (starting with r^0).

Reaching All Learners *Through Critical Thinking*

Display the following sequence:
3, 9, 27, 81, 243,…
Under each term, write a_1, a_2, a_3, a_4, and a_5. Write a_n directly below a_1 and ask how a_2 can be written in terms of n. a_{n+1} Then write the rule for finding the common ratio, r: divide each term by the one before it. Ask how the rule can be written algebraically. $r = \frac{a_{n+1}}{a_n}$ Show students that any two consecutive terms can be used; it can be the third and fourth terms or the fifth and sixth terms.

Mathematical Practices Questioning Strategies

EXAMPLE 1

- What makes the terms in some geometric sequences alternate signs?
- How do you find r when given part of a geometric sequence?

EXAMPLE 2

- When finding the nth term of a geometric sequence, why is r raised to $n - 1$, not n?
- How are geometric sequences and arithmetic sequences alike? How are they different?

Teaching Tip **Visual** Write the formula for a geometric sequence as *term wanted = first term* × *common ratio* (number of term wanted) − 1.

Power Presentations with PowerPoint®

Additional Examples

Example 3

A ball is dropped from a tower. The table shows the heights of the ball's bounces, which form a geometric sequence. What is the height of the 6th bounce?

9.375 cm

Bounce	Height (cm)
1	300
2	150
3	75

Also available online

Caution!

When writing a function rule for a sequence with a negative common ratio, remember to enclose r in parentheses.
$-2^{12} \neq (-2)^{12}$

C What is the 13th term of the geometric sequence 8, −16, 32, −64, … ?

8, −16, 32, −64

$\frac{-16}{8} = -2$ $\quad \frac{32}{-16} = -2$ $\quad \frac{-64}{32} = -2$ *The value of r is −2.*

$a_n = a_1 r^{n-1}$ *Write the formula.*

$a_{13} = 8(-2)^{13-1}$ *Substitute 8 for a_1, 13 for n, and −2 for r.*

$= 8(-2)^{12}$ *Simplify the exponent.*

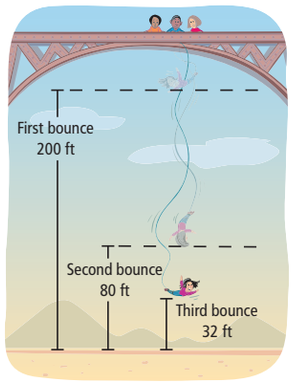
$= 32{,}768$ *Use a calculator.*

CHECK IT OUT! 2. What is the 8th term of the sequence 1000, 500, 250, 125, … ?

7.8125

EXAMPLE 3 Sports Application

A bungee jumper jumps from a bridge. The diagram shows the bungee jumper's height above the ground at the top of each bounce. The heights form a geometric sequence. What is the bungee jumper's height at the top of the 5th bounce?



200, 80, 32

$\frac{80}{200} = 0.4$ $\quad \frac{32}{80} = 0.4$

$a_n = a_1 r^{n-1}$ *Write the formula.*

$a_5 = 200(0.4)^{5-1}$ *Substitute 200 for a_1, 5 for n, and 0.4 for r.*

$= 200(0.4)^4$ *Simplify the exponent.*

$= 5.12$ *Use a calculator.*

The height of the 5th bounce is 5.12 feet.

CHECK IT OUT! 3. The table shows a car's value for 3 years after it is purchased. The values form a geometric sequence. How much will the car be worth in the 10th year?

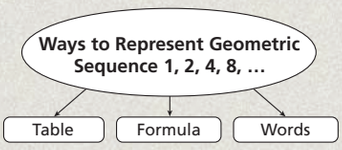
$1342.18

Year	Value ($)
1	10,000
2	8,000
3	6,400

MATHEMATICAL PRACTICES

THINK AND DISCUSS

1. How do you determine whether a sequence is geometric?
2. **GET ORGANIZED** Copy and complete the graphic organizer. In each box, write a way to represent the geometric sequence.



Know it! Note

MATHEMATICAL PRACTICES Questioning Strategies

EXAMPLE 3

- How can you tell whether the common ratio will be greater than one or less than one?
- How do you expect the sequence to continue? Why?

Teaching Tip **Math Background** The box shows the recursive definition of a geometric sequence; in this form, the nth term is determined using the terms that precede it. The formula is the iterative definition of a geometric sequence.

3 Close

Summarize

Tell students that they have learned about two kinds of sequences in this course: arithmetic and geometric. Arithmetic sequences have common differences and geometric sequences have common ratios. Common ratios are found by dividing each term by the term before it. Have students state in their own words how to find any term in a geometric sequence.

ONGOING ASSESSMENT and INTERVENTION

***Diagnose Before* the Lesson**
Warm Up, TE

***Monitor During* the Lesson**
Check It Out! Exercises, SE
Questioning Strategies, TE

***Assess After* the Lesson**
Lesson Quiz, TE
Alternative Assessment, TE

Answers to *Think and Discuss*

1. Possible answer: Divide each term after the first by the preceding term. If the quotients are all the same, the sequence is geometric.
2. See Additional Answers.

Learn It Online
Homework Help Online
Parent Resources Online

GUIDED PRACTICE

1. **Vocabulary** What is the *common ratio* of a geometric sequence? the value that each term is multiplied by to get the next term

SEE EXAMPLE 1

Find the next three terms in each geometric sequence.

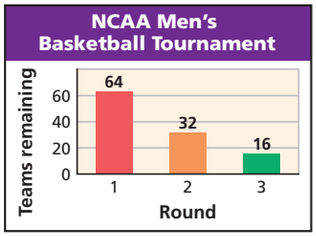
2. 2, 4, 8, 16, … 32, 64, 128
3. 400, 200, 100, 50, … 25, 12.5, 6.25
4. 4, −12, 36, −108, … 324, −972, 2916

SEE EXAMPLE 2

5. The first term of a geometric sequence is 1, and the common ratio is 10. What is the 10th term of the sequence? 1,000,000,000
6. What is the 11th term of the geometric sequence 3, 6, 12, 24, … ? 3072

SEE EXAMPLE 3

7. **Sports** In the NCAA men's basketball tournament, 64 teams compete in round 1. Fewer teams remain in each following round, as shown in the graph, until all but one team have been eliminated. The numbers of teams in each round form a geometric sequence. How many teams compete in round 5? 4



PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
8–13	1
14–15	2
16	3

Extra Practice
See Extra Practice for more Skills Practice and Applications Practice exercises.

Find the next three terms in each geometric sequence.

8. −2, 10, −50, 250, …
9. 32, 48, 72, 108, …
10. 625, 500, 400, 320, …
11. 6, 42, 294, …
12. 6, −12, 24, −48, …
13. 40, 10, $\frac{5}{2}$, $\frac{5}{8}$, …
14. The first term of a geometric sequence is 18 and the common ratio is 3.5. What is the 5th term of the sequence? 2701.125
15. What is the 14th term of the geometric sequence 1000, 100, 10, 1, … ? 0.0000000001, or 1×10^{-10}
16. **Physical Science** A ball is dropped from a height of 500 meters. The table shows the height of each bounce, and the heights form a geometric sequence. How high does the ball bounce on the 8th bounce? Round your answer to the nearest tenth of a meter. 83.9 m

Bounce	Height (m)
1	400
2	320
3	256

Find the missing term(s) in each geometric sequence.

17. 20, 40, 80, 160, …
18. 2, 6, 18, 54, …
19. 9, 3, 1, $\frac{1}{3}$, …
20. 3, 12, 48, 192, 768, …
21. 7, 1, $\frac{1}{7}$, $\frac{1}{49}$, $\frac{1}{343}$, …
22. 400, 100, 25, $\frac{25}{4}$, $\frac{25}{16}$, …
23. −3, 6, −12, 24, −48, …
24. $\frac{1}{9}$, ___, 1, −3, 9, … $-\frac{1}{3}$
25. 1, 17, 289, ___, … 4913

Determine whether each sequence could be geometric. If so, give the common ratio.

26. 2, 10, 50, 250, … yes; 5
27. 15, 5, $\frac{5}{3}$, $\frac{5}{9}$, … yes; $\frac{1}{3}$
28. 6, 18, 24, 38, … no
29. 9, 3, −1, −5, … no
30. 7, 21, 63, 189, … yes; 3
31. 4, 1, −2, −4, … no

9-1 Exercises

Assignment Guide

Assign *Guided Practice* exercises as necessary.

If you finished Examples 1–3
Basic 8–33, 34–39, 43–46
Average 8–40, 42–46, 54
Advanced 8–16 even, 17–54

Homework Quick Check
Quickly check key concepts.
Exercises: 8, 14, 16, 24, 32, 36

Teaching Tip **Diversity** **Exercise 7** is about a basketball tournament. Have students discuss the different types of tournaments they have participated in that have a similar type of elimination system.

Answers

8. −1250, 6250, −31,250
9. 162, 243, 364.5
10. 256, 204.8, 163.84
11. 2058, 14,406, 100,842
12. 96, −192, 384
13. $\frac{5}{32}$, $\frac{5}{128}$, $\frac{5}{512}$

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them. Exercises 7, 16, 32–33, 41, 43

Construct viable arguments and critique the reasoning of others. Exercises 40, 42

Look for and express regularity in repeated reasoning. Exercises 26–31, 34–39, 41, 51–54

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MULTI-STEP TEST PREP **Exercise 43** involves using a common ratio to find terms in a geometric sequence. This exercise prepares students for the Multi-Step Test Prep.

Answers

39. 12, 3, $\frac{3}{4}$, $\frac{3}{16}$

40. Each term is multiplied by 2^{n-1}, where n is the term number. For example, begin with the geometric sequence 4, 12, 36, 108, …, where $r = 3$. If r is doubled to 6, the sequence becomes 4, 24, 144, 864, ….

41a, 41b. See Additional Answers.

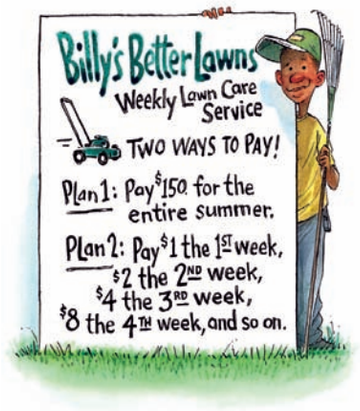
42. Divide each term by the term before it to find the value of r. Then use the formula $a_n = a_1 r^{n-1}$, where a_1 is the first term of the sequence.

54, 58–60. See Additional Answers.

32a. Yes, it is a geom. seq. with $r = 2$.

b. Possible answer: Plan 1; Under Plan 2, the cost for the 10th week alone is $512, which is more than the cost for the entire summer under Plan 1.

32. Multi-Step Billy earns money by mowing lawns for the summer. He offers two payment plans, as shown at right.



a. Do the payments for plan 2 form a geometric sequence? Explain.

b. If you were one of Billy's customers, which plan would you choose? (Assume that the summer is 10 weeks long.) Explain your choice.

33. Measurement When you fold a piece of paper in half, the thickness of the folded piece is twice the thickness of the original piece. A piece of copy paper is about 0.1 mm thick.

a. How thick is a piece of copy paper that has been folded in half 7 times? **1.28 cm**

b. Suppose that you could fold a piece of copy paper in half 12 times. How thick would it be? Write your answer in centimeters. **40.96 cm**

List the first four terms of each geometric sequence.

34. $a_1 = 3, a_n = 3(2)^{n-1}$ **3, 6, 12, 24**

35. $a_1 = -2, a_n = -2(4)^{n-1}$ **−2, −8, −32, −128**

36. $a_1 = 5, a_n = 5(-2)^{n-1}$ **5, −10, 20, −40**

37. $a_1 = 2, a_n = 2(2)^{n-1}$ **2, 4, 8, 16**

38. $a_1 = 2, a_n = 2(5)^{n-1}$ **2, 10, 50, 250**

39. $a_1 = 12, a_n = 12\left(\frac{1}{4}\right)^{n-1}$

40. Critical Thinking What happens to the terms of a geometric sequence when r is doubled? Use an example to support your answer.

41. Geometry The steps below describe how to make a geometric figure by repeating the same process over and over on a smaller and smaller scale.

Step 1 (stage 0) Draw a large square.

Step 2 (stage 1) Divide the square into four equal squares.

Step 3 (stage 2) Divide each small square into four equal squares.

Step 4 Repeat Step 3 indefinitely.

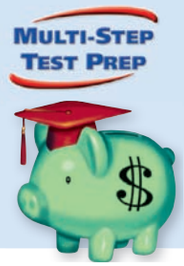
a. Draw stages 0, 1, 2, and 3.

b. How many small squares are in each stage? Organize your data relating stage and number of small squares in a table.

c. Does the data in part **b** form a geometric sequence? Explain. **yes; $r = 4$**

d. Write a rule to find the number of small squares in stage n. **$a_n = 4(4)^{n-1}$, or 4^n**

42. Write About It Write a series of steps for finding the nth term of a geometric sequence when you are given the first several terms.



43. a. Three years ago, the annual tuition at a university was $3000. The following year, the tuition was $3300, and last year, the tuition was $3630. If the tuition has continued to grow in the same manner, what is the tuition this year? What do you expect it to be next year? **$3993; $4392.30**

b. What is the common ratio? **1.1**

c. What would you predict the tuition was 4 years ago? How did you find that value? **$2727.27; divide tuition 3 years ago ($3000) by 1.1 (the common ratio).**

PRACTICE A

PRACTICE C

PRACTICE B

Find the next three terms in each geometric sequence.

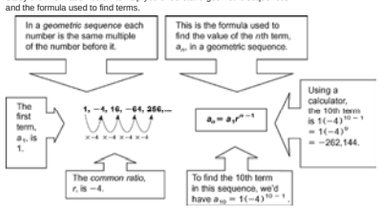
1. −5, −10, −20, −40, …
   −80; −160; −320
2. 7, 56, 448, 3584…
   28,672; 229,376; 1,835,008
3. −10, 40, −160, 640, …
   −2560, −10,240, −40,960
4. 40, 10, $\frac{5}{2}$, $\frac{5}{8}$, …
   $\frac{5}{32}, \frac{5}{128}, \frac{5}{512}$
5. The first term of a geometric sequence is 6 and the common ratio is −8. Find the 7th term.
   1,572,864
6. The first term of a geometric sequence is −3 and the common ratio is $\frac{1}{2}$. Find the 6th term.
   $-\frac{3}{32}$
7. The first term of a geometric sequence is −0.25 and the common ratio is −3. Find the 10th term.
   4920.75
8. What is the 12th term of the geometric sequence −4, −12, −36, …?
   −708,588
9. What is the 10th term of the geometric sequence 2, −6, 18, …?
   −39,366
10. What is the 6th term of the geometric sequence 50, 10, 2, …?
    0.016
11. A shoe store is discounting shoes each month. A pair of shoes cost $80. The table shows the discount prices for several months. Find the cost of the shoes after 8 months. Round your answer to the nearest cent.

Month	Price
1	$80.00
2	$72.00
3	$64.80

$38.26

READING STRATEGIES

Study the information below to help you understand geometric sequences and the formula used to find terms.



Complete the following about the geometric sequence 2, 6, 18, 54, … .

1. What is the first term, a_1? **2**
2. What is the common ratio, r? **3**
3. Describe how you found the common ratio in problem 2.
   Possible answer: I divided each term by the previous term.
4. What is the next term? **162**
5. Use the formula to find the 12th term. **354,294**

Complete the following.

6. Is 6, 9, 12, 15, … a geometric sequence? Explain.
   no; the ratios between terms (1.5, 1.$\overline{3}$, and 1.25) are different.
7. Find the 8th term in the geometric sequence 5, −10, 20, −40, … . **−640**
8. The first term of a geometric sequence is 14 and the common ratio is 6. Find the 5th term. **18,144**
9. Find the 9th term in the geometric sequence 256, 128, 64, 32, … . **1**

RETEACH

In a **geometric sequence**, each term is *multiplied* by the same number to get to the next term. This number is called the **common ratio**.

3 ×4 12 ×4 48 ×4 192 — The common ratio is 4.

Determine if the sequence 2, 6, 18, 54, … is a geometric sequence.

Divide each term by the term before it.

$\frac{54}{18} = 3$ $\frac{18}{6} = 3$ $\frac{6}{2} = 3$

This is a geometric sequence; 3 is the common ratio.

Determine if the sequence 5, 10, 15, 20, … is a geometric sequence.

Divide each term by the term before it.

$\frac{20}{15} = \frac{4}{3}$ $\frac{15}{10} = \frac{3}{2}$ $\frac{10}{5} = 2$

This is not a geometric sequence; there is no common ratio.

Find the next three terms in the geometric sequence 1, 4, 16, 64, ….

Step 1: Find the common ratio.

$\frac{64}{16} = 4$ $\frac{16}{4} = 4$ $\frac{4}{1} = 4$

Step 2: Continue to multiply by the common ratio.

$64 \times 4 = 256$ $256 \times 4 = 1024$ $1024 \times 4 = 4096$

The next three terms are 256, 1024, and 4096.

Determine if each sequence is a geometric sequence. Explain.

1. 2, 4, 6, 8, … **no; there is no common ratio.**
2. −4, 8, −16, 32, … **yes; the common ratio is −2.**
3. 32, 16, 8, 4, … **yes; the common ratio is $\frac{1}{2}$.**

Find the common ratio in each geometric sequence below. Then find the next three terms.

4. 1, 5, 25, 125, …
   5; 625, 3125, 15,625
5. −6, 12, −24, 48, …
   −2; −96, 192, −384
6. 4, 6, 9, 13.5, …
   1.5; 20.25, 30.375, 45.5625
7. $\frac{1}{4}, \frac{1}{2}, 1, 2, \ldots$
   2; 4, 8, 16

TEST PREP

44. Which of the following is a geometric sequence?

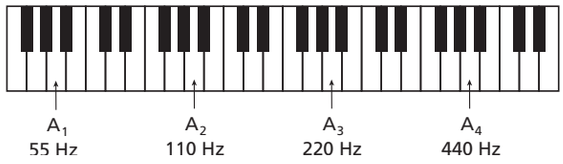
Ⓐ $\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ Ⓒ 3, 8, 13, 18, …

Ⓑ −2, −6, −10, −14, … Ⓓ 5, 10, 20, 40, …

45. Which equation represents the *n*th term in the geometric sequence 2, −8, 32, −128, …?

Ⓕ $a_n = (-4)^n$ Ⓖ $a_n = (-4)^{n-1}$ Ⓗ $a_n = 2(-4)^n$ Ⓙ $a_n = 2(-4)^{n-1}$

46. The frequency of a musical note, measured in hertz (Hz), is called its pitch. The pitches of the A keys on a piano form a geometric sequence, as shown.



What is the frequency of A_7?

Ⓐ 880 Hz Ⓑ 1760 Hz Ⓒ 3520 Hz Ⓓ 7040 Hz

CHALLENGE AND EXTEND

Find the next three terms in each geometric sequence.

47. $x, x^2, x^3, \ldots$ x^4, x^5, x^6

48. $2x^2, 6x^3, 18x^4, \ldots$ $54x^5, 162x^6, 486x^7$

49. $\frac{1}{y^3}, \frac{1}{y^2}, \frac{1}{y}, \ldots$ $1, y, y^2$

50. $\frac{1}{(x+1)^2}, \frac{1}{x+1}, 1, \ldots$ $x+1, (x+1)^2, (x+1)^3$

51. The 10th term of a geometric sequence is 0.78125. The common ratio is −0.5. Find the first term of the sequence. −400

52. The first term of a geometric sequence is 12 and the common ratio is $\frac{1}{2}$. Is 0 a term in this sequence? Explain.

52. No; each term of the seq. is found by multiplying the prev. term by the common ratio $\frac{1}{2}$. $\frac{1}{2}$ of any pos. number is always another pos. number.

53. A geometric sequence starts with 14 and has a common ration of 0.4. Colin finds that another number in the sequence is 0.057344. Which term in the sequence did Colin find? the 7th term

54. The first three terms of a sequence are 1, 2, and 4. Susanna said the 8th term of this sequence is 128. Paul said the 8th term is 29. Explain how the students found their answers. Why could these both be considered correct answers?

PROBLEM SOLVING

Write the correct answer.

1. A ball is dropped from 400 feet. The table shows the height of each bounce.

Bounce	Height (ft)
1	280
2	196
3	137.2

Find the height of the ball on the 6th bounce. Round your answer to the nearest tenth of a foot.

47.1 feet

2. A plant starts with 1 branch. Every year, each branch becomes 3 branches. A sketch of the plant for the first 3 years is shown. How many branches will the plant have in year 10?

19,683 branches

3. Jeanette started selling bagels to offices in her area. Her sales for the first 3 months are shown in the table.

Month	Sales ($)
1	$200.00
2	$230.00
3	$264.50

If this trend continues, find the amount of Jeanette's sales in Month 8.

$532

How many branches would the plant have in year 10 if the plant had 5 branches the first year? (Each branch still becomes 3 branches every year.)

98,415 branches

The table shows the number of houses in a new subdivision. Use the table to answer questions 4–7. Select the best answer.

Month	Houses
1	3
2	6
3	12
4	24

4. The number of houses forms a geometric sequence. What is *r*?

A 0.5 C 3

B 2 D 6

5. Assuming that the trend continues, how many houses would be in the subdivision in Month 6?

F 36 H 60

G 48 J 96

6. Management decides the subdivision is complete when the number of houses reaches 48. When will this happen?

A Month 5 C Month 7

B Month 6 D Month 8

7. Suppose the number of houses tripled every month. How many more houses would be in the subdivision in Month 4? (The number of houses in Month 1 is still 3.)

F 48 H 72

G 57 J 81

CHALLENGE

Series

A **sequence** is a list of numbers that often forms a pattern. You have worked with arithmetic sequences and geometric sequences.

A **series** is the sum of the terms of a sequence. You can either find a *partial sum* of a few terms or determine what happens when you sum infinitely many terms. (You'll look at infinite series in Algebra 2.)

The Greek capital letter sigma (Σ) is often used to represent summation in mathematics. Numbers above and below sigma tell you which terms to sum.

For example, $\sum_{n=1}^{5}$ represents the partial sum of the 1st to 5th terms.

Find the partial sums of these arithmetic series.

1. $\sum_{n=1}^{5} 4+(n-1)(3) = \frac{4}{n=1} + \frac{7}{n=2} + \frac{10}{n=3} + \frac{13}{n=4} + \frac{16}{n=5} = 50$
2. $\sum_{n=1}^{8} -10+(n-1)(2) = \frac{-10}{n=1} + \frac{-8}{n=2} + \frac{-6}{n=3} + \frac{-4}{n=4} + \frac{-2}{n=5} + \frac{0}{n=6} + \frac{2}{n=7} + \frac{4}{n=8} = -24$
3. $\sum_{n=1}^{6} 23+(n-1)(-5) = 23+18+13+8+3+(-2) = 63$
4. $\sum_{n=1}^{8} \frac{1}{2}+(n-1)\left(\frac{1}{8}\right) = \frac{1}{2}+\frac{5}{8}+\frac{3}{4}+\frac{7}{8} = \frac{11}{4}$ or $2\frac{3}{4}$
5. $\sum_{n=1}^{10} -7+(n-1)(-1.2) = -11.8+(-13)+(-14.2)+(-15.4)+(-16.6)+(-17.8) = -88.8$

Caution!

Find the partial sums of these geometric series.

6. $\sum_{n=1}^{5} 4(3)^{n-1} = \frac{4}{n=1} + \frac{12}{n=2} + \frac{36}{n=3} + \frac{108}{n=4} + \frac{324}{n=5} = 484$
7. $\sum_{n=1}^{8} -10(2)^{n-1} = \frac{-10}{n=1} + \frac{-20}{n=2} + \frac{-40}{n=3} + \frac{-80}{n=4} + \frac{-160}{n=5} + \frac{-320}{n=6} + \frac{-640}{n=7} + \frac{-1280}{n=8} = -2550$
8. $\sum_{n=1}^{6} 3(-2)^{n-1} = 3+(-6)+12+(-24)+48+(-96) = -63$
9. $\sum_{n=1}^{4} 16\left(\frac{1}{4}\right)^{n-1} = 16+4+1+\frac{1}{4} = \frac{85}{4}$ or $21\frac{1}{4}$
10. $\sum_{n=5}^{10} -6\left(-\frac{2}{3}\right)^{n-1} = -\frac{32}{27}+\frac{64}{81}+\left(-\frac{128}{243}\right)+\frac{256}{729}+\left(-\frac{512}{2187}\right)+\frac{1024}{6561} = \frac{-4256}{6561} = -0.649$

TEST PREP DOCTOR

The incorrect choices, **A, B,** and **C,** in **Exercise 44** are arithmetic sequences. They have common differences, not common ratios.

In **Exercise 45,** choices **F** and **G** can be quickly eliminated because they do not include the first term, 2. To help distinguish between **H** and **J,** have students substitute 1 for *n*, which will result in the first term.

Journal

Have students explain the difference between the rules used to create the following sequences:

3, 6, 9, 12, 15,…

3, 6, 12, 24, 48,…

ALTERNATIVE ASSESSMENT

Have students write the first five terms of a geometric sequence for which *r* is greater than one. Then have them write the first five terms of a geometric sequence for which *r* is less than one. Last, have them find the 10th and 20th terms of each of their sequences by using the formula taught in the lesson.

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9-1 Lesson Quiz

Find the next three terms in each geometric sequence.

1. 3, 15, 75, 375,…
1875, 9375, 46,875

2. $5, -1, \frac{1}{5}, -\frac{1}{25}, \frac{1}{125}, \ldots$
$-\frac{1}{625}, \frac{1}{3125}, -\frac{1}{15,625}$

3. The first term of a geometric sequence is 300 and the common ratio is 0.6. What is the 7th term of the sequence?
13.9968

4. What is the 15th term of the sequence 4, −8, 16, −32, 64,…? 65,536

5. The table shows a car's value for 3 years after it is purchased. The values form a geometric sequence. How much will the car be worth after 8 years? $5770.39

Year	Value ($)
1	18,000
2	15,300
3	13,005

Also available online

9-2 Organizer

Pacing: Traditional 1 day
Block $\frac{1}{2}$ day

Objectives: Evaluate exponential functions.

Identify and graph exponential functions.

Online Edition
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Countdown

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Warm Up

Simplify each expression. Round to the nearest whole number if necessary.

1. 3^2 9
2. 5^4 625
3. $2(3)^3$ 54
4. $\frac{2}{3}(3)^4$ 54
5. $-5(2)^5$ −160
6. $-\frac{1}{2}(4)^3$ −32
7. $100(0.5)^2$ 25
8. $3000(0.95)^8$ 1990

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Math Humor

Exponential Function: Why is my energy level falling?

Doctor: Your common ratio is below one. Your energy levels will rise if your common ratio is above one.

COMMON CORE

CC.9-12.F.IF.7 Graph functions expressed symbolically and show key features of the graph, by hand in simple cases and using technology for more complicated cases.*
e. Graph exponential … functions, showing intercepts and end behavior, ….
CC.9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.*
c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

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COMMON CORE

9-2 Exponential Functions

CC.9-12.F.IF.7e Graph exponential … functions, showing intercepts and end behavior, …. *Also* **CC.9-12.F.LE.1*, CC.9-12.F.IF.4*, CC.9-12.F.IF.8, CC.9-12.A.REI.10**

Objectives
Evaluate exponential functions.

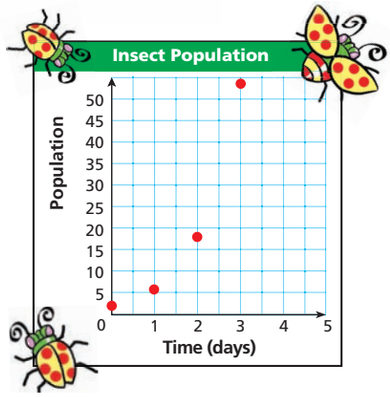
Identify and graph exponential functions.

Vocabulary
exponential function

Who uses this?
Scientists model populations with exponential functions.

The table and the graph show an insect population that increases over time.

Time (days)	Population	
0	2	
1	6	×3
2	18	×3
3	54	×3



A function rule that describes the pattern above is $f(x) = 2(3)^x$. This type of function, in which the independent variable appears in an exponent, is an **exponential function**. Notice that 2 is the starting population and 3 is the amount by which the population is multiplied each day.

Know it! Note

Exponential Functions

An exponential function has the form $f(x) = ab^x$, where $a \neq 0$, $b \neq 1$, and $b > 0$.

EXAMPLE 1 **Evaluating an Exponential Function**

A **The function $f(x) = 2(3)^x$ models an insect population after x days. What will the population be on the 5th day?**

$f(x) = 2(3)^x$ *Write the function.*
$f(5) = 2(3)^5$ *Substitute 5 for x.*
$= 2(243)$ *Evaluate 3^5.*
$= 486$ *Multiply.*

There will be 486 insects on the 5th day.

B **The function $f(x) = 1500(0.995)^x$, where x is the time in years, models a prairie dog population. How many prairie dogs will there be in 8 years?**

$f(x) = 1500(0.995)^x$
$f(8) = 1500(0.995)^8$ *Substitute 8 for x.*
≈ 1441 *Use a calculator. Round to the nearest whole number.*

There will be about 1441 prairie dogs in 8 years.

Helpful Hint
In Example 1B, round your answer to the nearest whole number because there can only be a whole number of prairie dogs.

CHECK IT OUT!

1. The function $f(x) = 8(0.75)^x$ models the width of a photograph in inches after it has been reduced by 25% x times. What is the width of the photograph after it has been reduced 3 times? **3.375 in.**

1 Introduce

EXPLORATION

Exponential Functions

A biologist is studying bacteria growth. He begins with 10 bacteria in a Petri dish. The number of bacteria doubles each hour.

1. Complete the middle column of the table to show the rate of bacteria growth.

Hour	Bacteria	Exponential Form
0	10	$10(2)^0$
1	20	$10(2)^1$
2	40	$10(2)^2$
3	80	$10(2)^3$
4	160	$10(2)^4$
5	320	$10(2)^5$

2. Complete the right-hand column of the table, which shows the number of bacteria written as a power of 2.
3. Look for a pattern in the table to help you write an equation for the number of bacteria B in terms of the number of hours h. $B = 10(2)^h$

THINK AND DISCUSS

4. **Explain** how to use your equation to find the number of bacteria after 15 hours. Substitute 15 for h; $B = 10(2)^{15} = 327{,}680$
5. **Describe** how your equation would be different if the biologist had started with 12 bacteria. $B = 12(2)^h$

Motivate

Write 6, 12, 24, 48 . . . on the board. Remind students that this is a geometric sequence with a common ratio of 2. Tell students that in this lesson they will study functions very similar to geometric sequences. But unlike geometric sequences, which have discrete terms, these new functions will be continuous. The graphs of these new functions will be unbroken.

Explorations and answers are provided in your online resources.

Remember that linear functions have constant first differences and quadratic functions have constant second differences. Exponential functions do not have constant differences, but they do have *constant ratios.*

As the x-values increase by a constant amount, the y-values are multiplied by a constant amount. This amount is the constant ratio and is the value of b in $f(x) = ab^x$.

x	$f(x) = 2(3)^x$	
1	6	
2	18	$\times 3$
3	54	$\times 3$
4	162	$\times 3$

$+1$, $+1$, $+1$

EXAMPLE 2 Identifying an Exponential Function

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

A $\{(-1, 1.5), (0, 3), (1, 6), (2, 12)\}$

x	y	
-1	1.5	
0	3	$\times 2$
1	6	$\times 2$
2	12	$\times 2$

$+1$, $+1$, $+1$

This is an exponential function. As the x-values increase by a constant amount, the y-values are multiplied by a constant amount.

B $\{(-1, -9), (1, 9), (3, 27), (5, 45)\}$

x	y	
-1	-9	
1	9	$\times (-1)$
3	27	$\times 3$
5	45	$\times \frac{5}{3}$

$+2$, $+2$, $+2$

This is *not* an exponential function. As the x-values increase by a constant amount, the y-values are *not* multiplied by a constant amount.

CHECK IT OUT! **Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.**

2a. $\{(-1, 1), (0, 0), (1, 1), (2, 4)\}$ **2b.** $\{(-2, 4), (-1, 2), (0, 1), (1, 0.5)\}$

2a. No; as the *x*-values change by a constant amount, the *y*-values are not multiplied by a constant amount.

2b. Yes; as the *x*-values change by a constant amount, the *y*-values are multiplied by a constant amount.

To graph an exponential function, choose several values of x (positive, negative, and 0) and generate ordered pairs. Plot the points and connect them with a smooth curve.

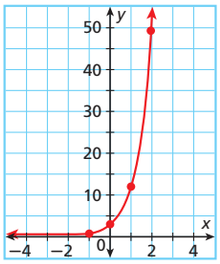
EXAMPLE 3 Graphing $y = ab^x$ with $a > 0$ and $b > 1$

Graph $y = 3(4)^x$.

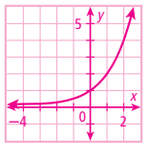
Choose several values of x and generate ordered pairs.

x	$y = 3(4)^x$
-1	0.75
0	3
1	12
2	48

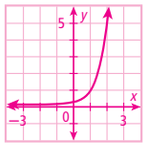
Graph the ordered pairs and connect with a smooth curve.



3a.



3b.



CHECK IT OUT! **3a.** Graph $y = 2^x$. **3b.** Graph $y = 0.2(5)^x$.

2 Teach

Guided Instruction

In **Example 1,** point out that b in an exponential function corresponds to r in a geometric sequence. Explain that these predictions are accurate only if all conditions remain the same throughout the given time period.

When graphing exponential functions, explain that $b = 1$ represents no change (no growth or decline—the graph is linear), so $b > 1$ represents growth and $b < 1$ represents decline.

Reaching All Learners

Through Graphic Organizers

To show why the exponent in a geometric sequence has 1 subtracted from the variable but an exponential function does not, create the following chart, which corresponds to **Example 1A.**

1st Term	2nd Term	3rd Term	4th Term	5th Term	6th Term
2	6	18	54	162	486
After 0 days	After 1 day	After 2 days	After 3 days	After 4 days	After 5 days

COMMON ERROR ALERT

When generating ordered pairs for exponential functions, students may multiply a by b and then raise that product to x. Remind students to use the correct order of operations.

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Additional Examples

Example 1

A. The function $f(x) = 500(1.035)^x$ models the amount of money in a certificate of deposit after x years. How much money will there be in 6 years? $614.63

B. The function $f(x) = 200{,}000(0.98)^x$, where x is the time in years, models the population of a city. What will the population be in 7 years? about 173,625

Example 2

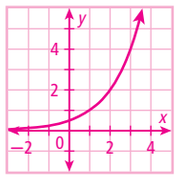
Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

A. $\{(0, 4), (1, 12), (2, 36), (3, 108)\}$ Yes; as the *x*-values change by a constant amount, the *y*-values are multiplied by a constant amount.

B. $\{(-1, -64), (0, 0), (1, 64), (2, 128)\}$ No; as the *x*-values change by a constant amount, the *y*-values are not multiplied by a constant amount.

Example 3

Graph $y = 0.5(2)^x$.



Also available online

MATHEMATICAL PRACTICES **Questioning Strategies**

EXAMPLES 1–2

- How do you know the exponent applies to b and not ab?

EXAMPLE 3

- What are the domain and range of an exponential function if both a and b are positive?

Teaching Tip **Transformations** The graph of $y = -ab^x$ is a reflection of $y = ab^x$ over the *x*-axis.

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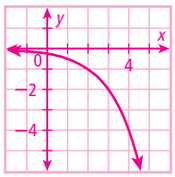
Additional Examples

Example 4

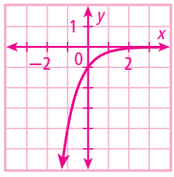
Graph $y = -\frac{1}{4}(2)^x$.



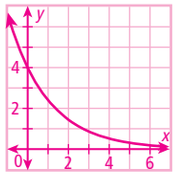
Example 5

Graph each exponential function.

A. $y = -1\left(\frac{1}{4}\right)^x$



B. $y = 4(0.6)^x$



Also available online

Mathematical Practices **Questioning Strategies**

EXAMPLES 4–5

- How can the values of *a* and *b* tell you which direction the graph takes and in which quadrants the graph is located?

Teaching Tip **Inclusion** Point out to students that, unlike linear functions, *b* in an exponential function is not the *y*-intercept.

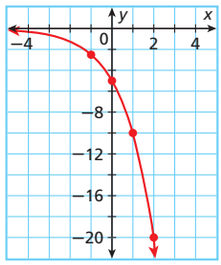
EXAMPLE 4 Graphing $y = ab^x$ with $a < 0$ and $b > 1$

Graph $y = -5(2)^x$.

Choose several values of x and generate ordered pairs.

x	$y = -5(2)^x$
−1	−2.5
0	−5
1	−10
2	−20

Graph the ordered pairs and connect with a smooth curve.



CHECK IT OUT!

4a. Graph $y = -6^x$.

4b. Graph $y = -3(3)^x$.

EXAMPLE 5 Graphing $y = ab^x$ with $0 < b < 1$

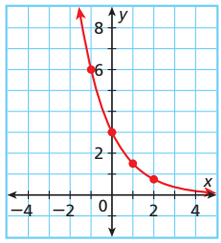
Graph each exponential function.

A $y = 3\left(\frac{1}{2}\right)^x$

Choose several values of x and generate ordered pairs.

x	$y = 3\left(\frac{1}{2}\right)^x$
−1	6
0	3
1	1.5
2	0.75

Graph the ordered pairs and connect with a smooth curve.

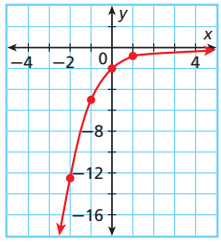


B $y = -2(0.4)^x$

Choose several values of x and generate ordered pairs.

x	$y = -2(0.4)^x$
−2	−12.5
−1	−5
0	−2
1	−0.8

Graph the ordered pairs and connect with a smooth curve.



CHECK IT OUT!

Graph each exponential function.

5a. $y = 4\left(\frac{1}{4}\right)^x$

5b. $y = -2(0.1)^x$

Teacher to Teacher

As students study exponential functions, it is critical that they understand why the base $b > 1$ makes the graph of $y = ab^x$ (where *a* is positive) increase while the base $0 < b < 1$ makes it decrease.

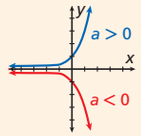
Students need plenty of real-world examples to help them understand the connection between the common ratio and the base *b*. For example, you can discuss what happens if you multiply something over and over by 5, as in $y = 5^x$, versus multiplying it over and over by $\frac{1}{5}$, as in $y = \left(\frac{1}{5}\right)^x$.

Once students understand how the base *b* affects the graph of $y = ab^x$, where $a > 0$, they can more easily remember how $a < 0$ affects the graph of $y = ab^x$ by thinking of it as a reflection across the *x*-axis.

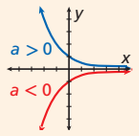
Jennifer Bauer
East Haven, CT

The box summarizes the general shapes of exponential function graphs.

Graphs of Exponential Functions



For $y = ab^x$, if $b > 1$, then the graph will have one of these shapes.



For $y = ab^x$, if $0 < b < 1$, then the graph will have one of these shapes.

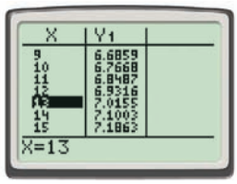
EXAMPLE 6 Statistics Application

In the year 2000, the world population was about 6 billion, and it was growing by 1.21% each year. At this growth rate, the function $f(x) = 6(1.0121)^x$ gives the population, in billions, x years after 2000. Using this model, in about what year does the population reach 7 billion?

Enter the function into the **Y=** *editor of a graphing calculator.*

Caution!

The function values give the population *in billions*, so a *y*-value of 7 means 7 billion.



Press **2nd** **GRAPH** (TABLE) *. Use the arrow keys to find a y-value as close to 7 as possible. The corresponding x-value is 13.*

The world population reaches 7 billion in about 2013.

CHECK IT OUT!

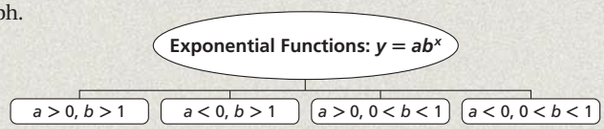
6. An accountant uses $f(x) = 12{,}330(0.869)^x$, where x is the time in years since the purchase, to model the value of a car. When will the car be worth \$2000? **after about 13 yr**

MATHEMATICAL PRACTICES

THINK AND DISCUSS

1. How can you find the constant ratio of a set of exponential data?

2. GET ORGANIZED Copy and complete the graphic organizer. In each box, give an example of an appropriate exponential function and sketch its graph.



COMMON ERROR ALERT

Students might use the word *slope* when describing exponential functions. Remind them that slope, when referring to linear functions, refers to a constant rate of change.

Teaching Tip **Multiple Representations** In **Example 6,** students could also graph the function and trace as close to $y = 7$ as possible.

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Additional Examples

Example 6

In 2000, each person in India consumed an average of 13 kg of sugar. Sugar consumption in India is projected to increase by 3.6% per year. At this growth rate, the function $f(x) = 13(1.036)^x$ gives the average yearly amount of sugar, in kilograms, consumed per person x years after 2000. Using this model, in about what year does sugar consumption average about 18 kg per person? 2009

Also available online

MATHEMATICAL PRACTICES **Questioning Strategies**

EXAMPLE 6

- Why is the *x*-value not the answer to the question?
- What types of conditions could change this prediction?

3 Close

Summarize

Write $f(x) = ab^x$ on the board. Point out the initial amount a, the common ratio b, and the independent variable x. Remind students that the independent variable always serves as the exponent in an exponential function. Then review the different shapes of exponential graphs. If a is positive, then the range is all positive numbers greater than 0, with an increasing graph for values of b greater than one. If a is negative, the graph is reflected across the *x*-axis.

ONGOING ASSESSMENT and INTERVENTION

***Diagnose* *Before* the Lesson**
Warm Up, TE

***Monitor* *During* the Lesson**
Check It Out! Exercises, SE
Questioning Strategies, TE

***Assess* *After* the Lesson**
Lesson Quiz, TE
Alternative Assessment, TE

Answers to *Think and Discuss*

1. Possible answer: Make a table of values. Use *x*-values that change by the same amount each time as you move down the column. Then divide each *y*-value, starting with the second row, by the *y*-value before it. The quotient is the common ratio.
2. See Additional Answers.

Assignment Guide

Assign *Guided Practice* exercises as necessary.

If you finished Examples **1–2**
- **Basic** 18–24, 38–42
- **Average** 18–24, 35, 38–42
- **Advanced** 18–24, 35, 38–42

If you finished Examples **1–6**
- **Basic** 18–34, 38–43, 45–47
- **Average** 18–34 even, 35–47, 51
- **Advanced** 18–34 even, 35–51

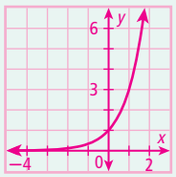
Homework Quick Check
Quickly check key concepts.
Exercises: 18, 22, 26, 28, 32, 34, 40

Answers

3. As the *x*-values change by a constant amount, the *y*-values are not multiplied by a constant amount.

4. As the *x*-values change by a constant amount, the *y*-values are multiplied by a constant amount.

5.



6–16. See Additional Answers.

21–24. For explanations, see Additional Answers.

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them. Exercises 2, 17–20, 34–36, 43

Construct viable arguments and critique the reasoning of others. Exercises 37, 44

Use appropriate tools strategically. Exercises 38–39

Look for and make use of structure. Exercise 51

Look for and express regularity in repeated reasoning. Exercise 47

Learn It Online
State Resources Online

9-2 Exercises

Learn It Online
Homework Help Online
Parent Resources Online

GUIDED PRACTICE

1. **Vocabulary** Tell whether $y = 3x^4$ is an *exponential function*. Explain your answer. No; there is no variable in the exponent.

SEE EXAMPLE 1

2. **Physics** The function $f(x) = 50{,}000(0.975)^x$, where x represents the underwater depth in meters, models the intensity of light below the water's surface in lumens per square meter. What is the intensity of light 200 meters below the surface? Round your answer to the nearest whole number. 316 lumens/m²

SEE EXAMPLE 2

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

3. $\{(-1, -1), (0, 0), (1, -1), (2, -4)\}$ no

4. $\{(0, 1), (1, 4), (2, 16), (3, 64)\}$ yes

Graph each exponential function.

SEE EXAMPLE 3

5. $y = 3^x$

6. $y = 5^x$

7. $y = 10(3)^x$

8. $y = 5(2)^x$

SEE EXAMPLE 4

9. $y = -2(3)^x$

10. $y = -4(2)^x$

11. $y = -3(2)^x$

12. $y = 2(3)^x$

SEE EXAMPLE 5

13. $y = -\left(\frac{1}{4}\right)^x$

14. $y = \left(\frac{1}{3}\right)^x$

15. $y = 2\left(\frac{1}{4}\right)^x$

16. $y = -2(0.25)^x$

SEE EXAMPLE 6

17. The function $f(x) = 57.8(1.02)^x$ gives the number of passenger cars, in millions, in the United States x years after 1960. Using this model, in about what year does the number of passenger cars reach 200 million? about 2023

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
18–20	1
21–24	2
25–27	3
28–30	4
31–33	5
34	6

Extra Practice
See Extra Practice for more Skills Practice and Applications Practice exercises.

18. **Sports** If a golf ball is dropped from a height of 27 feet, the function $f(x) = 27\left(\frac{2}{3}\right)^x$ gives the height in feet of each bounce, where x is the bounce number. What will be the height of the 4th bounce? $5\frac{1}{3}$ ft

19. Suppose the depth of a lake can be described by the function $y = 334(0.976)^x$, where x represents the number of weeks from today. Today, the depth of the lake is 334 ft. What will the depth be in 6 weeks? Round your answer to the nearest whole number. 289 ft

20. **Physics** A ball rolling down a slope travels continuously faster. Suppose the function $y = 1.3(1.41)^x$ describes the speed of the ball in inches per minute. How fast will the ball be rolling in 15 minutes? Round your answer to the nearest hundredth. 225.02 in./min

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

21. $\left\{(-2, 9), (-1, 3), (0, 1), \left(1, \frac{1}{3}\right)\right\}$ yes

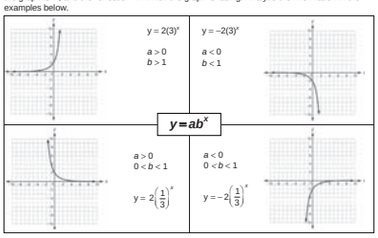
22. $\{(-1, 0), (0, 1), (1, 4), (2, 9)\}$ no

23. $\{(-1, -5), (0, -3), (1, -1), (2, 1)\}$ no

24. $\{(-3, 6.25), (-2, 12.5), (-1, 25), (0, 50)\}$ yes

READING STRATEGIES

An *exponential function* is always in the form $f(x) = ab^x$. And the graph of an exponential function is always the same general shape. But the values of a and b determine the quadrants the graph is in, and the "direction" in which the graph is facing. Analyze the information in the examples below.



Respond to each statement by writing *sometimes*, *always*, or *never*.

1. When a is negative, the graph of $y = ab^x$ is contained entirely in quadrants III and IV. always
2. When b is a fraction, the graph of $y = ab^x$ is contained entirely in quadrants I and II. sometimes
3. The y-intercept of $y = ab^x$ is at a. always
4. The graph of $y = ab^x$ is a straight line. never
5. Complete the table of values and sketch the graph of $y = -3(2)^x$.

x	y
−1	$-\frac{3}{2}$
0	−3
1	−6
2	−12

RETEACH

An **exponential function** has the independent variable as the exponent.
$y = 3^x$ and $y = -2(0.5)^x$ are exponential functions.

A set of ordered pairs satisfies an exponential function if the y-values are multiplied by a constant amount as the x-values change by a constant amount.

Tell whether the following ordered pairs satisfy an exponential function.

x	y	
3	4	Think 4 × ? = 12.
5	12	Think 12 × ? = 36.
7	36	Think 36 × ? = 108.
9	108	

The x-values increase by the constant amount 2.
Each y-value is multiplied by the constant amount 3.
This is an exponential function.

x	y	
1	2	Think 2 × ? = 4.
2	4	Think 4 × ? = 6.
3	6	Think 6 × ? = 8.
4	8	

The x-values increase by the constant amount 1.
The y-value is multiplied by 2, then 1.5, Then $1.\overline{3}$. There is no constant ratio.
This is not an exponential function.

The population of a school can be described by the function $f(x) = 1500(1.02)^x$, where x represents the number of years since the school was built. What will be the population of the school in 12 years?

$f(x) = 1500(1.02)^x$
$f(12) = 1500(1.02)^{12}$ *Substitute 12 for x.*
≈ 1902 *Round number of people to the nearest whole number.*

Tell whether the ordered pairs satisfy an exponential function.

1.

x	y
−1	1.5
−2	3
−3	6
−4	12

yes

2.

x	y
1	1
2	2
3	6
4	24

no

3.

x	y
−2	−2
−1	−10
0	−50
1	−250

yes

5. A population of pigs is expected to increase at a rate of 4% each year. If the original population is 1000, the function $f(x) = 1000(1.04)^x$ gives the population in x years. What will be the population in 12 years? ≈1601

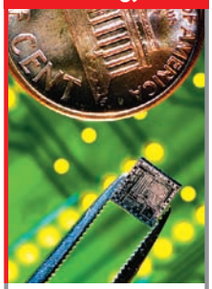
Graph each exponential function.

25. $y = 1.5^x$ **26.** $y = \frac{1}{3}(3)^x$ **27.** $y = 100(0.7)^x$

28. $y = -2(4)^x$ **29.** $y = -1(5)^x$ **30.** $y = -\frac{1}{2}(4)^x$

31. $y = 4\left(\frac{1}{2}\right)^x$ **32.** $y = -2\left(\frac{1}{3}\right)^x$ **33.** $y = 0.5(0.25)^x$

LINK Technology



Early silicon chips were about the size of your pinky finger and held one transistor. Today, chips the size of a baby's fingernail hold over 100 million transistors.

34. Technology Moore's law states that the maximum number of transistors that can fit on a silicon chip doubles every two years. The function $f(x) = 42(1.41)^x$ models the number of transistors, in millions, that can fit on a chip, where x is the number of years since 2000. Using this model, in what year can a chip hold 1 billion transistors? **about 2009**

35. Multi-Step A computer randomly creates three different functions. The functions are $y = (3.1x + 7)^2$, $y = 4.8(2)^x$, and $y = \frac{1}{5}(6)^x$. The computer then generates the y value 38.4. Given the three different functions, determine which one is exponential *and* produces the generated number. **$y = 4.8(2)^x$**

36. Contests As a promotion, a clothing store draws the name of one of its customers each week. The prize is a coupon for the store. If the winner is not present at the drawing, he or she cannot claim the prize, and the amount of the coupon increases for the following week's drawing. The function $f(x) = 20(1.2)^x$ gives the amount of the coupon in dollars after x weeks of the prize going unclaimed.

a. What is the amount of the coupon after 2 weeks of the prize going unclaimed? **$28.80**

36b. after 9 weeks

b. After how many weeks of the prize going unclaimed will the amount of the coupon be greater than $100?

c. What is the original amount of the coupon? **$20**

d. Find the percent increase each week. **20%**

37. Critical Thinking In the definition of exponential function, the value of b cannot be 1, and the value of a cannot be 0. Why?

37. If the value of b were 1, the function would be constant. If the value of a were 0, the function would be the constant function $y = 0$.

Graphing Calculator **Graph each group of functions on the same screen. How are their graphs alike? How are they different?**

38. $y = 2^x$, $y = 3^x$, $y = 4^x$ **39.** $y = \left(\frac{1}{2}\right)^x$, $y = \left(\frac{1}{3}\right)^x$, $y = \left(\frac{1}{4}\right)^x$

Evaluate each of the following for the given value of x.

40. $f(x) = 4^x$; $x = 3$ **64** **41.** $f(x) = -(0.25)^x$; $x = 1.5$ **−0.125** **42.** $f(x) = 0.4(10)^x$; $x = -3$ **0.0004, or 4×10^{-4}**

MULTI-STEP TEST PREP

43. a. The annual tuition at a community college since 2001 is modeled by the equation $C = 2000(1.08)^n$, where C is the tuition cost and n is the number of years since 2001. What was the tuition cost in 2001? **$2000**

b. What is the annual percentage of tuition increase? **8%**

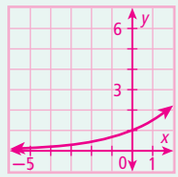
c. Find the tuition cost in 2006. **$2938.66**

Lucidio Studio, Inc./Alamy

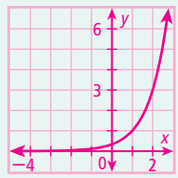
MULTI-STEP TEST PREP **Exercise 43** involves evaluating exponential functions. This exercise prepares students for the Multi-Step Test Prep.

Answers

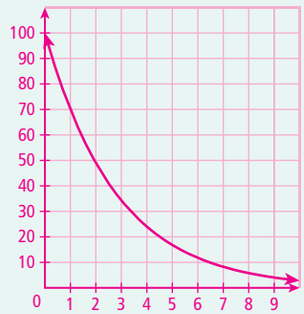
25.



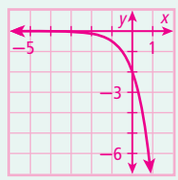
26.



27.



28.



29–33, 38–39. See Additional Answers.

PRACTICE A

PRACTICE C

PRACTICE B

1. If a basketball is bounced from a height of 15 feet, the function $f(x) = 15(0.75)^x$ gives the height of the ball in feet of each bounce, where x is the bounce number. What will be the height of the 5th bounce? Round to the nearest tenth of a foot. **3.6 feet**

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

2. {(2, 4), (4, 8), (6, 16), (8, 32)} **yes, as the x-values increase by 2, the y values are multiplied by 2.**

3. {(−2, 5), (−1, 10), (0, 15), (1, 20)} **no, as the x-values increase by 1, the y-values are not multiplied by a constant amount.**

4. {(1, 750), (2, 150), (3, 30), (4, 6)} **yes, as the x-values increase by 1, the y-values are multiplied by $\frac{1}{5}$**

5. $\left\{\left(-5, \frac{1}{3}\right), (0, 1), (5, 3), (10, 9)\right\}$ **yes, as the x-values increase by 5, the y-values are multiplied by 3.**

Graph each exponential function.

6. $y = 5(2)^x$ 7. $y = -2(3)^x$ 8. $y = 3\left(\frac{1}{2}\right)^x$

In the year 2000, the population of Virginia was about 7,400,000. Between the years 2000 and 2004, the population in Virginia grew at a rate of 5.4%. At this growth rate, the function $f(x) = 7{,}400{,}000\ (1.054)^x$ gives the population x years after 2000.

9. In what year will the population reach 15,000,000? **2014**

10. In what year will the population reach 20,000,000? **2019**

PROBLEM SOLVING

Write the correct answer.

1. The function $f(x) = 6\ (1.5)^x$ models the length of a photograph in inches after the photo has been enlarged by 50% x times.
   a. What is the length of the photograph after it has been enlarged 4 times? **30.375 in.**
   b. Graph the function.

2. A population of 550 rabbits is increasing by 2.5% each year. The function $y = 5.5\ (1.025)^x$ gives the population of rabbits, in hundreds, x years from now. About how long will it take the population to reach 600 rabbits? 1200 rabbits? **4 years** **32 years**

3. The function $y = 200\ (1.0004)^x$ models the balance on a customer's line of credit x days after the end of the grace period (the time when no interest accumulates). If Jack does not make any payments, determine Jack's balance 30 days after the grace period ends. **$202.41**

4. The function $f(x) = 2300\ (0.995)^x$ models enrollment in a high school, where x is the number of years after 2005. Use the model to estimate the enrollment in 2013. **2210**

A lake was stocked with fish in early April. Select the best answer.

5. The function $f(x) = 300(0.85)^x$ models the number of landlocked salmon in the lake x months after the lake was stocked. Which is the best estimate of the number of landlocked salmon in early July?
   A 157 C 217
   B 184 D 255

6. The function $f(x) = 75(1.2)^x$ models the number of rainbow trout in the lake x years after 2005. Likewise, the function $f(x) = 105\ (1.08)^x$ models the number of perch in the lake for the same time period. Which statement is NOT true?
   A The population of perch will reach 120 before the population of rainbow trout does.
   B In 2009, the number of rainbow trout will exceed the number of perch.
   C In 2007, the number of rainbow trout will be less than 100.
   D In 10 years, there will be at least twice as many rainbow trout as perch.

7. The function $f(x) = 400\ (1.05)^x$ models the number of bass x months after the lake was stocked. During what month will the population reach 600?
   F September H November
   G October **J** December

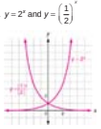
CHALLENGE

9-2 *Exploring Reciprocals and Reflections*

The diagram at right shows a triangle and its reflection across the y-axis. Notice that the triangle and its reflection are congruent figures.

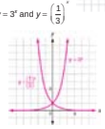
Graph each pair of functions on the grid provided. Describe how the bases of the exponential functions are related. Then describe how each graph is related to the other.

1. $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$



The bases are reciprocals. The graphs are reflections across the y-axis.

2. $y = 3^x$ and $y = \left(\frac{1}{3}\right)^x$



The bases are reciprocals. The graphs are reflections across the y-axis.

3. Let b be a number greater than 1. Consider $y = b^x$ and $y = \left(\frac{1}{b}\right)^x$. Write a statement that describes how the bases are related and how the graphs are related.
   The bases are reciprocals of one another and the graphs are reflections of one another across the y-axis.

Explain why the graphs in each pair are reflections of one another across the y-axis.

4. $y = 10^x$ and $y = \left(\frac{1}{10}\right)^x$
   Because 10 and $\frac{1}{10}$ are reciprocals

5. $y = 5^x$ and $y = 0.2^x$
   Because $0.2 = \left(\frac{1}{5}\right)$ and 5 and $\frac{1}{5}$ are reciprocals

6. $y = 4^x$ and $y = 4^{-x}$
   Because $4^{-x} = \left(\frac{1}{4}\right)^x$ and 4 and $\frac{1}{4}$ are reciprocals

7. $y = \left(\frac{5}{4}\right)^x$ and $y = 1.25^{-x}$
   Because $1.25^{-x} = \left(\frac{5}{4}\right)^{-x} = \left(\frac{4}{5}\right)^x$ and $\frac{5}{4}$ and $\frac{4}{5}$ are reciprocals

TEST PREP DOCTOR Students who chose **A** or **B** in **Exercise 45** may have forgotten that the range of an exponential function is restricted to either values greater than 0 or values less than 0. Students who chose **D** may have forgotten that the domain of an exponential function is all real numbers.

Students who chose **H** in **Exercise 46** raised 1.4 to the power of $x - 1$.

Journal

Have students describe two ways of identifying an exponential function.

ALTERNATIVE ASSESSMENT

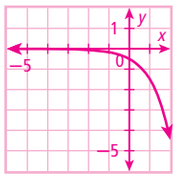
Sketch the four basic shapes of exponential graphs on the board. Then have students write an exponential function that would have a graph with a similar shape for each one.

Power Presentations with PowerPoint®

9-2 Lesson Quiz

Tell whether each set of ordered pairs satisfies an exponential function. Explain your answer.

1. $\{(0, 0), (1, -2), (2, -16), (3, -54)\}$ No; for a constant change in x, y is not multiplied by the same value.
2. $\{(0, -5), (1, -2.5), (2, -1.25), (3, -0.625)\}$ Yes; for a constant change in x, y is multiplied by the same value.
3. Graph $y = -0.5(3)^x$.



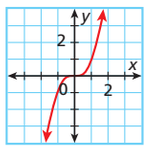
4. The function $y = 11.6(1.009)^x$ models residential energy consumption in quadrillion Btu where x is the number of years after 2003. What will residential energy consumption be in 2013? ≈12.7 quadrillion Btu
5. In 2000, the population of Texas was about 21 million, and it was growing by about 2% per year. At this growth rate, the function $f(x) = 21(1.02)^x$ gives the population, in millions, x years after 2000. Using this model, in about what year will the population reach 30 million? 2018

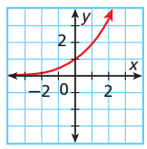
Also available online

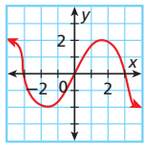
44. **Write About It** Your employer offers two salary plans. With plan A, your salary is $f(x) = 10{,}000(2x)$, where x is the number of years you have worked for the company. With plan B, your salary is $g(x) = 10{,}000(2)^x$. Which plan would you choose? Why?

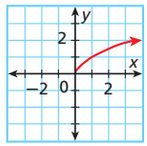
TEST PREP

45. Which graph shows an exponential function?

(A)


(C)


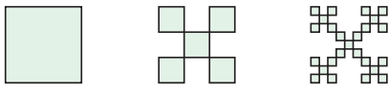
(B)


(D)


46. The function $f(x) = 15(1.4)^x$ represents the area in square inches of a photograph after it has been enlarged x times by a factor of 140%. What is the area of the photograph after it has been enlarged 4 times?

(F) 5.6 square inches
(G) 57.624 square inches
(H) 41.16 square inches
(J) 560 square inches

47. Look at the pattern. How many squares will there be in the nth stage?



(A) $5n$ (B) $2.5 \cdot 2^n$ (C) 25^{n-1} (D) 5^n

CHALLENGE AND EXTEND

Solve each equation.

48. $4^x = 64$ 3
49. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$ 3
50. $2^x = \frac{1}{16}$ −4

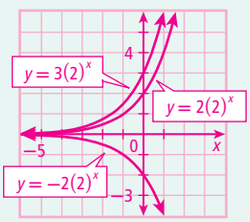
51. Graph the following functions: $y = 2(2)^x$, $y = 3(2)^x$, $y = -2(2)^x$. Then make a conjecture about the relationship between the value of a and the y-intercept of $y = ab^x$.

Answers

44. Possible answer: The following table shows how much money you could earn with each plan.

Year	Salary w/ Plan A ($)	Salary w/ Plan B ($)
0	0	10,000
1	20,000	20,000
2	40,000	40,000
3	60,000	80,000

I would choose plan B because plan A doesn't pay you anything for the first year and because after 3 years, plan B pays more money.

51.


The value of a is the y-intercept.

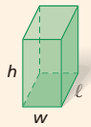
Changing Dimensions

What happens to the volume of a three-dimensional figure when you repeatedly double the dimensions?

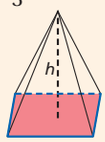
Recall these formulas for the volumes of common three-dimensional figures.

Cube $V = s^3$

Rectangular Prism $V = \ell wh$



Pyramid $V = \frac{1}{3}(\text{area of base}) \cdot h$

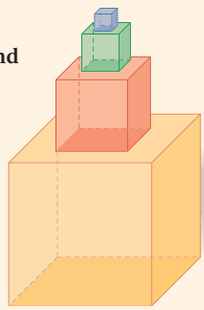


Base

Changing the dimensions of three-dimensional figures results in geometric sequences.

Example

Find the volume of a cube with a side length of 3 cm. Double the side length and find the new volume. Repeat two more times. Show the patterns for the side lengths and volumes as geometric sequences. Identify the common ratios.

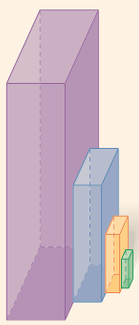


Cube	Side Length (cm)	Volume (cm^3)
1	3	27
2	6 (×2)	216 (×8)
3	12 (×2)	1,728 (×8)
4	24 (×2)	13,824 (×8)

The side lengths and the volumes form geometric sequences. The sequence of the side lengths has a common ratio of 2. The sequence of the volumes has a common ratio of 2^3, or 8.

The patterns in the example above are a specific instance of a general rule.

> When the dimensions of a solid figure are multiplied by x, the volume of the figure is multiplied by x^3.



Try This

1. The large rectangular prism at right is 8 in. wide, 16 in. long, and 32 in. tall. The dimensions are multiplied by $\frac{1}{2}$ to create each next smaller prism. Show the patterns for the dimensions and the volumes as geometric sequences. Identify the common ratios.
2. A pyramid has a height of 8 cm and a square base of 3 cm on each edge. Triple the dimensions two times. Show the patterns for the dimensions and the volumes as geometric sequences. Identify the common ratios.

Answers

1. widths: 8, 4, 2, 1; common ratio: $\frac{1}{2}$;
   lengths: 16, 8, 4, 2; common ratio: $\frac{1}{2}$
   heights: 32, 16, 8, 4; common ratio: $\frac{1}{2}$,
   volumes: 4096, 512, 64, 8; common ratio: $\frac{1}{8}$
2. heights: 8, 24, 72; common ratio: 3
   edge of bases: 3, 9, 27; common ratio: 3
   volumes: 24, 648, 17,496; common ratio: 27

Organizer

Pacing:
Traditional $\frac{1}{2}$ day
Block $\frac{1}{4}$ day

Objective: Express with geometric sequences how the changing dimensions of solid figures affects volume.

Online Edition

Teach

Remember

Students review and apply volume formulas for common solids.

Visual Ask students what they notice about the sets of figures on the lesson pages. If no one realizes it, point out that each set shows a group of similar figures.

Close

Assess

The edges of a cube are multiplied by 5. What happens to the volume of the cube? The volume is multiplied by $5^3 = 125$.

Learn It Online
State Resources Online

HOUGHTON MIFFLIN HARCOURT

Quiz for Lessons 9-1 and 9-2

9-1 Geometric Sequences

Find the next three terms in each geometric sequence.

1. 3, 6, 12, 24, … **48, 96, 192**
2. −1, 2, −4, 8, … **−16, 32, −64**
3. −2400, −1200, −600, −300, … **−150, −75, −37.5**
4. The first term of a geometric sequence is 2 and the common ratio is 3. What is the 8th term of the sequence? **4374**

9-2 Exponential Functions

5. The function $f(x) = 3(1.1)^x$ gives the length (in inches) of an image after being enlarged by 10% x times. What is the length of the image after it has been enlarged 4 times? Round your answer to the nearest hundredth. **4.39 in.**

Graph each exponential function.

6. $y = 3^x$
7. $y = 2(2)^x$
8. $y = -2(4)^x$
9. $y = -(0.5)^x$
10. The function $f(x) = 40(0.8)^x$ gives the amount of a medication in milligrams present in a patient's system x hours after taking a 40-mg dose. In how many hours will there be less than 2 mg of the drug in a patient's system? **14 h**

READY TO GO ON? SECTION 9A

Organizer

Objective: Assess students' mastery of concepts and skills in this section.

Resources

Assessment Resources
Section Quiz

INTERVENTION

Resources

***Ready to Go On? Intervention and Enrichment* Worksheets**

***Ready to Go On?* Online**

Answers

6–9 See Additional Answers.

READY TO GO ON?

Diagnose and Prescribe

NO INTERVENE

Ready to Go On? Intervention	*Ready to Go On? Intervention*, Section 9A: Worksheets	Online	
Lesson 9-1	9-1 Intervention	Activity 9-1	Diagnose and Prescribe Online
Lesson 9-2	9-2 Intervention	Activity 9-2	

YES ENRICH

***Ready to Go On? Enrichment*, Section 9A**
Worksheets
Online

SECTION 9B

Functions, Models, and Patterns

One-Minute Section Planner

Lesson	Lab Resources	Materials
Algebra Lab Model Growth and Decay • Fold and cut paper to model exponential growth and decay. ☐ SAT-10 ☐ NAEP ☐ ACT ☑ SAT ☑ SAT Subject Tests	***Algebra Lab Activities*** Lab Recording Sheet	**Required** notebook paper, scissors
Lesson 9-3 Exponential Growth and Decay • Solve problems involving exponential growth and decay. ☐ SAT-10 ☑ NAEP ☐ ACT ☑ SAT ☑ SAT Subject Tests	***Algebra Lab Activities*** Algebra Lab	**Required** graphing calculator
Extension Patterns and Recursion • Identify and extend patterns using recursion. ☐ SAT-10 ☐ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests		
Lesson 9-4 Linear, Quadratic, and Exponential Models • Compare linear, quadratic, and exponential models. • Given a set of data, decide which type of function models the data and write an equation to describe the function. ☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests		**Optional** graphing calculator
Extension Linear and Nonlinear Rates of Change • Identify linear and nonlinear rates of change. • Compare rates of change. ☐ SAT-10 ☐ NAEP ☐ ACT ☐ SAT ☐ SAT Subject Tests		
Lesson 9-5 Comparing Functions • Compare functions in different representations. • Estimate and compare rates of change. ☑ SAT-10 ☑ NAEP ☑ ACT ☑ SAT ☑ SAT Subject Tests		**Required** graph paper **Optional** Graphing Calculator

Note: If NAEP is checked, the content is tested on either the Grade 8 or the Grade 12 NAEP assessment.

MK = *Manipulatives Kit*

Selected Answers in Student Edition

The answers to the following exercises are included in the Selected Answers section at the back of the Student Edition textbook.

Lesson 3: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 37, 39, 45, 47, 49, 51, 53

Lesson 4: 1, 3, 5, 7, 11, 13, 15, 17, 19, 21, 27, 29

Lesson 5: 1, 3, 5, 7, 13, 15a, 15b, 15c

All answers for the Check it Out! exercises in each lesson and for the Chapter Study Guide: Review are included in the Student Edition.

Math Background

COMMON CORE PROFESSIONAL DEVELOPMENT

EXPONENTIAL GROWTH AND DECAY

Lesson 9-3

Exponential growth and decay are important applications of exponential functions.

Exponential growth occurs when a quantity increases by the same rate in each time period. Thus, exponential growth can be understood as an extension of percent change in which a percent increase is repeatedly applied. *Exponential decay* occurs when a quantity decreases by the same rate in each time period, and, similar to exponential growth, can be understood as an extension of percent change in which a percent decrease is repeatedly applied.

In an exponential growth situation with initial amount a ($a > 0$) and rate of growth r expressed as a decimal, after one time period, the new amount is $a + ar$, or $a(1 + r)$. After the second time period, the new amount is $a(1 + r)(1 + r)$, or $a(1 + r)^2$. Continuing in this way shows that after t time periods, the final amount y is given by $y = a(1 + r)^t$. For exponential decay, a similar argument shows that the final amount y after t time periods is given by $y = a(1 - r)^t$.

An important attribute of exponential growth and decay is the fact that the amount added or subtracted in each time period is proportional to the amount already present. For exponential growth, this means that as the amount becomes greater, the amount of increase in each time period also becomes greater. Contrast this to linear growth, in which the amount of increase remains constant.

COMPOUND INTEREST

Lesson 9-3

Compound interest is closely related to exponential growth. In fact, for interest compounded once per year, the formula is analogous to that for exponential growth: $A = P(1 + r)^t$, where P represents the principal and A represents the total balance after t years. If interest is compounded n times per year, then the rate must be divided by n, and because there are nt periods in t years, the formula becomes $A = P\left(1 + \frac{r}{n}\right)^{nt}$.

As n increases, we approach a situation in which interest is compounded continuously. In future courses, students will study continuously compounded interest, which is given by $A = Pe^{rt}$ where $e \approx 2.71828$.

Organizer

Use with the lesson Exponential Growth and Decay

Pacing:
Traditional 1 day
Block $\frac{1}{2}$ day

Objective: Fold and cut paper to model exponential growth and decay.

Materials: notebook paper, scissors

Online Edition

Resources

Algebra Lab Activities
Lab Recording Sheet

Teach

Discuss

Remind students that $2^0 = 1$ and $2^{-n} = \frac{1}{2^n}$.

In **Activity 1,** the exponent equals the number of folds. In **Activity 2,** the exponent equals the opposite of the number of cuts.

Close

Key Concept

Multiplying repeatedly by numbers greater than 1 causes a quantity to increase, while multiplying repeatedly by numbers between 0 and 1 causes a quantity to decrease.

Assessment

Journal For each activity, have students explain what b represents in the exponential function $f(x) = b^x$.

MATHEMATICAL PRACTICES

Look for and express regularity in repeated reasoning.

COMMON CORE

CC.9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).*

CC.9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.*

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State Resources Online

9-3 Algebra LAB

Model Growth and Decay

You can fold and cut paper to model quantities that increase or decrease exponentially.

Use with Exponential Growth and Decay

MATHEMATICAL PRACTICES **Look for and express regularity in repeated reasoning.**

CC.9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).* *Also* **CC.9-12.F.LE.1***

Activity 1

1. Copy the table at right.
2. Fold a piece of notebook paper in half. Then open it back up. Count the number of regions created by the fold. Record your answer in the table.
3. Now fold the paper in half twice. Record the number of regions created by the folds in the table.
4. Repeat this process for 3, 4, and 5 folds.

Folds	Regions
0	1
1	2
2	4
3	8
4	16
5	32

Try This

1. When the number of folds increases by 1, the number of regions ___?___. doubles
2. For each row of the table, write the number of regions as a power of 2. $2^0, 2^1, 2^2, 2^3, 2^4, 2^5$
3. Write an exponential expression for the number of regions formed by n folds. 2^n
4. If you could fold the paper 8 times, how many regions would be formed? 256
5. How many times would you have to fold the paper to make 512 regions? 9

Activity 2

1. Copy the table at right.
2. Begin with a square piece of paper. The area of the paper is 1 square unit. Cut the paper in half. Each piece has an area of $\frac{1}{2}$ square unit. Record the result in the table.
3. Cut one of those pieces in half again, and record the area of one of the new, smaller pieces in the table.
4. Repeat this process for 3, 4, and 5 cuts.

Cuts	Area
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
4	$\frac{1}{16}$
5	$\frac{1}{32}$

Try This

6. When the number of cuts increases by 1, the area ___?___. is divided in half
7. For each row of the table, write the area as a power of 2. $2^0, 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, 2^{-5}$
8. Write an exponential expression for the area after n cuts. 2^{-n}
9. What would be the area after 7 cuts? $\frac{1}{128}$
10. How many cuts would you have to make to get an area of $\frac{1}{256}$ square unit? 8

COMMON CORE

9-3 Exponential Growth and Decay

CC.9-12.F.LE.2 Construct linear and exponential functions … given a graph, a description of a relationship, or two input-output pairs … * *Also* **CC.9-12.F.LE.5***, **CC.9-12.F.LE.1***, **CC.9-12.F.BF.1***

Objective
Solve problems involving exponential growth and decay.

Vocabulary
exponential growth
compound interest
exponential decay
half-life

Why learn this?
Exponential growth and decay describe many real-world situations, such as the value of artwork. (See Example 1.)

Exponential growth occurs when a quantity increases by the same rate r in each time period t. When this happens, the value of the quantity at any given time can be calculated as a function of the rate and the original amount.

Know it! Note

Exponential Growth

An exponential growth function has the form $y = a(1 + r)^t$, where $a > 0$.

y represents the final amount.

a represents the original amount.

r represents the rate of growth expressed as a decimal.

t represents time.

EXAMPLE 1 **Exponential Growth**

The original value of a painting is $1400, and the value increases by 9% each year. Write an exponential growth function to model this situation. Then find the value of the painting in 25 years.

Step 1 Write the exponential growth function for this situation.

$y = a(1 + r)^t$ *Write the formula.*

$= 1400(1 + 0.09)^t$ *Substitute 1400 for a and 0.09 for r.*

$= 1400(1.09)^t$ *Simplify.*

Step 2 Find the value in 25 years.

$y = 1400(1.09)^t$

$= 1400(1.09)^{25}$ *Substitute 25 for t.*

$\approx 12{,}072.31$ *Use a calculator and round to the nearest hundredth.*

The value of the painting in 25 years is $12,072.31.

Helpful Hint
In Example 1, round to the nearest hundredth because the problem deals with money. This means you are rounding to the nearest cent.

CHECK IT OUT!

1. A sculpture is increasing in value at a rate of 8% per year, and its value in 2000 was $1200. Write an exponential growth function to model this situation. Then find the sculpture's value in 2006. $y = 1200(1.08)^t$; $1904.25

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9-3 Organizer

Pacing: Traditional 2 days
Block 1 day

Objective: Solve problems involving exponential growth and decay.

Algebra Lab
In *Algebra Lab Activities*

Online Edition
Tutorial Videos, Interactivity

Power Presentations with PowerPoint®

Warm Up

Simplify each expression.

1. $(4 + 0.05)^2$ 16.4025
2. $25(1 + 0.02)^3$ 26.5302
3. $1 + \frac{0.03}{4}$ 1.0075
4. The first term of a geometric sequence is 3 and the common ratio is 2. What is the 5th term of the sequence? 48
5. The function $f(x) = 2(4)^x$ models an insect population after x days. What is the insect population after 3 days? 128 insects

Also available online

Math Humor

Q: Why was the banker's son sent to the principal's office?

A: He had an interest in another student's test paper.

COMMON CORE

CC.9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).*

CC.9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.*
c. Recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.

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State Resources Online

1 Introduce

EXPLORATION

Exponential Growth and Decay

Kendrick puts $200 in a savings account that earns 4% interest each year.

1. How much interest will Kendrick earn in the first year? $8
2. What is the total amount of money that Kendrick will have at the end of the first year? $208
3. In the second year, Kendrick will receive 4% interest on the total amount of money you found in Problem 2. How much interest will that be? $8.32
4. Complete the table.

Year	Process: Beginning Amount × 1.04	Exponential Form	Total ($)
1	200(1.04)	$200(1.04)$	208.00
2	[200(1.04)](1.04)	$200(1.04)^2$	216.32
3	[200(1.04)(1.04)](1.04)	$200(1.04)^3$	224.97
4	[200(1.04)(1.04)(1.04)](1.04)	$200(1.04)^4$	233.97
5	[200(1.04)(1.04)(1.04)(1.04)](1.04)	$200(1.04)^5$	243.33

THINK AND DISCUSS

5. **Describe** any patterns you see in the table.
6. **Show** how to write an equation for the total amount of money A that Kendrick will have in year t. $A = 200(1.04)^t$

5. The total amount of money in year t is $200(1.04)^t$.

Motivate

Review the formula for simple interest I: $I = Prt$, where P is the principal, or original amount, r is the interest rate, and t is time.

Have students compute the simple interest on $200 invested at 4% for 20 years. $160

Explorations and answers are provided in your online resources.

Power Presentations with PowerPoint®

Additional Examples

Example 1

The original value of a painting is $9000 and the value increases by 7% each year. Write an exponential growth function to model this situation. Then find the painting's value in 15 years.

$y = 9000(1.07)^t$; $24,831.28

Example 2

Write a compound interest function to model each situation. Then find the balance after the given number of years.

A. $1200 invested at a rate of 2% compounded quarterly; 3 years $A = 1200(1.005)^{4t}$; $1274.01

B. $15,000 invested at a rate of 4.8% compounded monthly; 2 years. $A = 15{,}000(1.004)^{12t}$; $16,508.22

Also available online

Questioning Strategies

EXAMPLE 1

- How is the formula for exponential growth similar to the standard form of an exponential function?
- How is this formula like that for finding the nth term of a geometric sequence?

EXAMPLE 2

- How is compound interest different from simple interest?
- In the formula for compound interest, why is r divided by n? Why is t multiplied by n?

Teaching Tip **Reading Math** Have students find different definitions of *compound*. Examples may include a substance formed by two or more ingredients, to combine, or to make stronger and more intense. Relate these definitions to the meaning of *compound interest*.

ENGLISH LANGUAGE LEARNERS

A common application of exponential growth is *compound interest*. Recall that simple interest is earned or paid only on the principal. **Compound interest** is interest earned or paid on *both* the principal and previously earned interest.

Know it! Note

Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

- A represents the balance after t years.
- P represents the principal, or original amount.
- r represents the annual interest rate expressed as a decimal.
- n represents the number of times interest is compounded per year.
- t represents time in years.

EXAMPLE 2 ***Finance Application***

Write a compound interest function to model each situation. Then find the balance after the given number of years.

A **$1000 invested at a rate of 3% compounded quarterly; 5 years**

Step 1 Write the compound interest function for this situation.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ *Write the formula.*

$= 1000\left(1 + \frac{0.03}{4}\right)^{4t}$ *Substitute 1000 for P, 0.03 for r, and 4 for n.*

$= 1000(1.0075)^{4t}$ *Simplify.*

Step 2 Find the balance after 5 years.

$A = 1000(1.0075)^{4(5)}$ *Substitute 5 for t.*

$= 1000(1.0075)^{20}$

≈ 1161.18 *Use a calculator and round to the nearest hundredth.*

The balance after 5 years is $1161.18.

Reading Math

For compound interest,
- *annually* means "once per year" ($n = 1$).
- *quarterly* means "4 times per year" ($n = 4$).
- *monthly* means "12 times per year" ($n = 12$).

B **$18,000 invested at a rate of 4.5% compounded annually; 6 years**

Step 1 Write the compound interest function for this situation.

$A = P\left(1 + \frac{r}{n}\right)^{nt}$ *Write the formula.*

$= 18{,}000\left(1 + \frac{0.045}{1}\right)^{t}$ *Substitute 18,000 for P, 0.045 for r, and 1 for n.*

$= 18{,}000(1.045)^{t}$ *Simplify.*

Step 2 Find the balance after 6 years.

$A = 18{,}000(1.045)^{6}$ *Substitute 6 for t.*

$\approx 23{,}440.68$ *Use a calculator and round to the nearest hundredth.*

The balance after 6 years is $23,440.68.

CHECK IT OUT!

Write a compound interest function to model each situation. Then find the balance after the given number of years.

2a. $A = 1200(1.00875)^{4t}$; $1379.49

2a. $1200 invested at a rate of 3.5% compounded quarterly; 4 years

2b. $A = 4000(1.0025)^{12t}$; $5083.47

2b. $4000 invested at a rate of 3% compounded monthly; 8 years

2 Teach

Guided Instruction

Show that the formulas for exponential growth and decay are just variations of $y = ab^x$ with either $1 + r$ or $1 - r$ replacing b. Point out also the similarities between $y = ab^x$ and the compound interest formula: P is the original amount a, and $1 + \frac{r}{n}$ is b. Explain that the exponent must be the number of time periods nt and that the common ratio must correspond with it. Because r is always given as an annual rate, it must be divided by n when n is not 1.

Exponential decay occurs when a quantity decreases by the same rate r in each time period t. Just like exponential growth, the value of the quantity at any given time can be calculated by using the rate and the original amount.

Know it! Note — Exponential Decay

An exponential decay function has the form $y = a(1 - r)^t$, where $a > 0$.

- y represents the final amount.
- a represents the original amount.
- r represents the rate of decay as a decimal.
- t represents time.

Notice an important difference between exponential growth functions and exponential decay functions. For exponential growth, the value inside the parentheses will be greater than 1 because r is added to 1. For exponential decay, the value inside the parentheses will be less than 1 because r is subtracted from 1.

EXAMPLE 3 Exponential Decay

The population of a town is decreasing at a rate of 1% per year. In 2000 there were 1300 people. Write an exponential decay function to model this situation. Then find the population in 2008.

Step 1 Write the exponential decay function for this situation.

$y = a(1 - r)^t$ *Write the formula.*

$= 1300(1 - 0.01)^t$ *Substitute 1300 for a and 0.01 for r.*

$= 1300(0.99)^t$ *Simplify.*

Step 2 Find the population in 2008.

$y = 1300(0.99)^8$ *Substitute 8 for t.*

≈ 1200 *Use a calculator and round to the nearest whole number.*

The population in 2008 is approximately 1200 people.

Helpful Hint

In Example 3, round your answer to the nearest whole number because there can only be a whole number of people.

CHECK IT OUT!

3. The fish population in a local stream is decreasing at a rate of 3% per year. The original population was 48,000. Write an exponential decay function to model this situation. Then find the population after 7 years. $y = 48{,}000(0.97)^t$; 38,783

A common application of exponential decay is *half-life*. The **half-life** of a substance is the time it takes for one-half of the substance to decay into another substance.

Know it! Note — Half-life

$A = P(0.5)^t$

- A represents the final amount.
- P represents the original amount.
- t represents the number of half-lives in a given time period.

COMMON ERROR ALERT

When writing compound interest functions, students may forget to multiply t by n. Remind students that the exponent is the number of times the interest is compounded, which is not necessarily the number of years.

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Additional Examples

Example 3

The population of a town is decreasing at a rate of 3% per year. In 2000, there were 1700 people. Write an exponential decay function to model this situation. Then find the population in 2012. $y = 1700(0.97)^t$; 1180

Also available online

MATHEMATICAL PRACTICES Questioning Strategies

EXAMPLE 3

- How is the formula for exponential decay similar to the formula for exponential growth? How is it different?

Reaching All Learners

Through Modeling

Have students complete the table for 80 g of a substance for 45 h, given that the half-life of the substance is 15 h.

Time passed (h)	Substance left (g)
15	half of 80 = 40
30	half of 40 = 20
45	half of 20 = 10

Then show the formula: $y = 80(0.5)^3$. Point out that the exponent 3 represents the number of rows in the table.

Through Critical Thinking

Have students complete the following:

$800 \times (1 - 0.55) = 800 \times 0.45 = 360$

$800 \times (1 - 0.05) = 800 \times 0.95 = 760$

$800 \times 1 = 800$

$800 \times (1 + 0.45) = 800 \times 1.45 = 1160$

$800 \times (1 + 0.95) = 800 \times 1.95 = 1560$

Point out that answers are less than 800 for $1 - r$ and greater than 800 for $1 + r$. Therefore, r is added to 1 in growth problems and subtracted from 1 in decay problems.

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Additional Examples

Example 4

Astatine-218 has a half-life of 2 seconds.

a. Find the amount left from a 500-gram sample of astatine-218 after 10 seconds. 15.625 g

b. Find the amount left from a 500-gram sample of astatine-218 after 1 minute. 0.00000047 g

Also available online

Mathematical Practices — Questioning Strategies

EXAMPLE 4

- How does the formula for half-life compare to $y = ab^x$? Why is b always 0.5?
- How do you find the value of t in the half-life formula?

EXAMPLE 4 Science Application

Fluorine-20 has a half-life of 11 seconds.

A **Find the amount of fluorine-20 left from a 40-gram sample after 44 seconds.**

Step 1 Find t, the number of half-lives in the given time period.

$\frac{44 \text{ s}}{11 \text{ s}} = 4$ *Divide the time period by the half-life. The value of t is 4.*

Step 2 $A = P(0.5)^t$ *Write the formula.*

$= 40(0.5)^4$ *Substitute 40 for P and 4 for t.*

$= 2.5$ *Use a calculator.*

There are 2.5 grams of fluorine-20 remaining after 44 seconds.

B **Find the amount of fluorine-20 left from a 40-gram sample after 2.2 minutes. Round your answer to the nearest hundredth.**

Step 1 Find t, the number of half-lives in the given time period.

$2.2(60) = 132$ *Find the number of seconds in 2.2 minutes.*

$\frac{132 \text{ s}}{11 \text{ s}} = 12$ *Divide the time period by the half-life. The value of t is $\frac{132}{11} = 12$.*

Step 2 $A = P(0.5)^t$ *Write the formula.*

$= 40(0.5)^{12}$ *Substitute 40 for P and 12 for t.*

≈ 0.01 *Use a calculator. Round to the nearest hundredth.*

There is about 0.01 gram of fluorine-20 remaining after 2.2 minutes.

CHECK IT OUT!

4a. Cesium-137 has a half-life of 30 years. Find the amount of cesium-137 left from a 100-milligram sample after 180 years. **1.5625 mg**

4b. Bismuth-210 has a half-life of 5 days. Find the amount of bismuth-210 left from a 100-gram sample after 5 weeks. (*Hint:* Change 5 weeks to days.) **0.78125 g**

Mathematical Practices

THINK AND DISCUSS

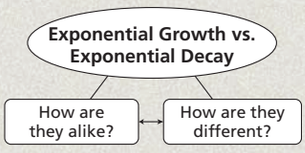
1. Describe three real-world situations that can be described by exponential growth or exponential decay functions.

2. The population of a town after t years can be modeled by $P = 1000(1.02)^t$. Is the population increasing or decreasing? By what percentage rate?

3. An exponential function is a function of the form $y = ab^x$. Explain why both exponential growth functions and exponential decay functions are exponential functions.

Know it! Note

4. GET ORGANIZED Copy and complete the graphic organizer.



3 Close

Summarize

Write $y = ab^x$ on the board. Below it, write and identify by name the four formulas covered in the lesson. Have students compare each one with the standard form of an exponential function, identifying a and b. Have them also identify each as growth or decay.

ONGOING ASSESSMENT and INTERVENTION

***Diagnose Before* the Lesson**
Warm Up, TE

***Monitor During* the Lesson**
Check It Out! Exercises, SE
Questioning Strategies, TE

***Assess After* the Lesson**
Lesson Quiz, TE
Alternative Assessment, TE

Answers to *Think and Discuss*

1. Possible answers: interest earned on an investment, population growth or decline, radioactive decay
2. increasing; by 2% per year
3. An exponential growth function has the form $y = a(1 + r)^t$. The base $(1 + r)$ corresponds to the base b. The exponent t corresponds to the exponent x. An exponential decay function has the form $y = a(1 - r)^t$. The base $(1 - r)$ corresponds to the base b. The exponent t corresponds to the exponent x.
4. See Additional Answers.

9-3 Exercises

Learn It Online
Homework Help Online
Parent Resources Online

GUIDED PRACTICE

1. **Vocabulary** The function $y = 0.68(2)^x$ is an example of _____?_____. (*exponential growth* or *exponential decay*) exponential growth

SEE EXAMPLE 1

Write an exponential growth function to model each situation. Then find the value of the function after the given amount of time.

2. The cost of tuition at a college is $12,000 and is increasing at a rate of 6% per year; 4 years. $y = 12{,}000\,(1.06)^t$; $15,149.72
3. The number of student-athletes at a local high school is 300 and is increasing at a rate of 8% per year; 5 years. $y = 300(1.08)^t$; 441

SEE EXAMPLE 2

Write a compound interest function to model each situation. Then find the balance after the given number of years.

4. $1500 invested at a rate of 3.5% compounded annually; 4 years $A = 1500(1.035)^t$; $1721.28
5. $4200 invested at a rate of 2.8% compounded quarterly; 6 years $A = 4200(1.007)^{4t}$; $4965.43

SEE EXAMPLE 3

Write an exponential decay function to model each situation. Then find the value of the function after the given amount of time.

6. The value of a car is $18,000 and is depreciating at a rate of 12% per year; 10 years. $y = 18{,}000\,(0.88)^t$; $5013.02
7. The amount (to the nearest hundredth) of a 10-mg dose of a certain antibiotic decreases in your bloodstream at a rate of 16% per hour; 4 hours. $y = 10(0.84)^t$; 4.98 mg

SEE EXAMPLE 4

8. Bismuth-214 has a half-life of approximately 20 minutes. Find the amount of bismuth-214 left from a 30-gram sample after 1 hour. 3.75 g
9. Mendelevium-258 has a half-life of approximately 52 days. Find the amount of mendelevium-258 left from a 44-gram sample after 156 days. 5.5 g

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
10–13	1
14–17	2
18–19	3
20	4

Extra Practice

See Extra Practice for more Skills Practice and Applications Practice exercises.

Write an exponential growth function to model each situation. Then find the value of the function after the given amount of time.

10. Annual sales for a company are $149,000 and are increasing at a rate of 6% per year; 7 years. $y = 149{,}000\,(1.06)^t$; $224,040.91
11. The population of a small town is 1600 and is increasing at a rate of 3% per year; 10 years. $y = 1600(1.03)^t$; 2150
12. A new savings account starts at $700 and increases at 1.2% yearly; 8 years. $A = 700(1.012)^t$; $770.09
13. Membership of a local club grows at a rate of 7.8% yearly and currently has 30 members; 6 years. $A = 30(1.078)^t$; 47 members

Write a compound interest function to model each situation. Then find the balance after the given number of years.

14. $28,000 invested at a rate of 4% compounded annually; 5 years $A = 28{,}000(1.04)^t$; $34,066.28
15. $7000 invested at a rate of 3% compounded quarterly; 10 years $A = 7000(1.0075)^{4t}$; $9438.44
16. $3500 invested at a rate of 1.8% compounded monthly; 4 years $A = 3500(1.0015)^{12t}$; $3761.09
17. $12,000 invested at a rate of 2.6% compounded annually; 15 years

17. $A = 12{,}000(1.026)^t$; $17,635.66

9-3 Exercises

Assignment Guide

Assign *Guided Practice* exercises as necessary.

If you finished Examples 1–2

Basic 10–17, 29, 30
Average 10–17, 29, 30
Advanced 10–17, 29, 37

If you finished Examples 1–4

Basic 10–20, 29–33, 35, 38, 39, 41, 45–48
Average 10–20, 22–34 even, 35–48, 50–54 even
Advanced 10–20, 22–34 even, 36–54

Homework Quick Check

Quickly check key concepts.
Exercises: 11, 15, 19, 20, 25, 31, 40

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them. Exercises 2–20, 29–34, 37–39, 48–54

Construct viable arguments and critique the reasoning of others. Exercises 35–36, 42–44

Use appropriate tools strategically. Exercise 41

Attend to precision. Exercise 40

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State Resources Online

Teaching Tip **Communicating Math** In **Exercise 34,** help students find t by writing the exponent as $\frac{\text{time period}}{\text{half-life}}$. Then they can substitute 3500 for the numerator and 5700 for the denominator. Show students that the answer is reasonable by considering that one half-life is not yet finished. Therefore, half of the 15 grams is not yet gone.

Teaching Tip **Science Link** **Exercise 34** involves carbon-14 dating, one common method of dating ancient objects. Scientists also use other isotopes to date objects. Some isotopes' half-lives are millions of years!

Write an exponential decay function to model each situation. Then find the value of the function after the given amount of time.

18. The population of a town is 18,000 and is decreasing at a rate of 2% per year; 6 years. **$y = 18{,}000(0.98)^t$; 15,945**

19. The value of a book is \$58 and decreases at a rate of 10% per year; 8 years. **$y = 58(0.9)^t$; \$24.97**

20. The half-life of bromine-82 is approximately 36 hours. Find the amount of bromine-82 left from an 80-gram sample after 6 days. **$y = 80(0.5)^t$; 5 g**

Identify each of the following functions as exponential growth or decay. Then give the rate of growth or decay as a percent.

21. $y = 3(1.61)^t$ **growth; 61%**

22. $y = 39(0.098)^t$ **decay; 90.2%**

23. $y = a\left(\frac{2}{3}\right)^t$ **decay; $33\frac{1}{3}$%**

24. $y = a\left(\frac{3}{2}\right)^t$ **growth; 50%**

25. $y = a(1.1)^t$ **growth; 10%**

26. $y = a(0.8)^t$ **decay; 20%**

27. $y = a\left(\frac{5}{4}\right)^t$ **growth; 25%**

28. $y = a\left(\frac{1}{2}\right)^t$ **decay; 50%**

Write an exponential growth or decay function to model each situation. Then find the value of the function after the given amount of time.

29. The population of a country is 58,000,000 and grows by 0.1% per year; 3 years. **$y = 58{,}000{,}000(1.001)^t$; 58,174,174**

30. An antique car is worth \$32,000, and its value grows by 7% per year; 5 years. **$y = 32{,}000(1.07)^t$; \$44,881.66**

31. An investment of \$8200 loses value at a rate of 2% per year; 7 years. **$y = 8200(0.98)^t$; \$7118.63**

32. A new car is worth \$25,000, and its value decreases by 15% each year; 6 years. **$y = 25{,}000(0.85)^t$; \$9428.74**

33. The student enrollment in a local high school is 970 students and increases by 1.2% per year; 5 years. **$y = 970(1.012)^t$; 1030**

34. Archaeology Carbon-14 dating is a way to determine the age of very old organic objects. Carbon-14 has a half-life of about 5700 years. An organic object with $\frac{1}{2}$ as much carbon-14 as its living counterpart died 5700 years ago. In 1999, archaeologists discovered the oldest bridge in England near Testwood, Hampshire. Carbon dating of the wood revealed that the bridge was 3500 years old. Suppose that when the bridge was built, the wood contained 15 grams of carbon-14. How much carbon-14 would it have contained when it was found by the archaeologists? Round to the nearest hundredth. **9.80 g**

A computer-generated image of what the bridge at Testwood might have looked like

Tom Goskar/Wessex Archaeology Ltd.

35. ERROR ANALYSIS Two students were asked to find the value of a \$1000-item after 3 years. The item was depreciating (losing value) at a rate of 40% per year. Which is incorrect? Explain the error.

B; possible answer: student B did not subtract the rate from 1.

A	B
$1000(0.6)^3$	$1000(0.4)^3$
\$216	\$64

36. Critical Thinking The value of a certain car can be modeled by the function $y = 20{,}000(0.84)^t$, where t is time in years. Will the value ever be zero? Explain.

36. No; possible answer: there is no value for t that would make $(0.84)^t$ equal 0.

37. The value of a rare baseball card increases every year at a rate of 4%. Today, the card is worth \$300. The owner expects to sell the card as soon as the value is over \$600. How many years will the owner wait before selling the card? Round your answer to the nearest whole number. **18 years**

PRACTICE A
PRACTICE C
PRACTICE B

Write an exponential growth function to model each situation. Then find the value of the function after the given amount of time.

1. Annual sales for a fast food restaurant are \$650,000 and are increasing at a rate of 4% per year; 5 years. **$y = 650{,}000(1.04)^x$ ≈\$790,824.39**
2. The population of a school is 800 students and is increasing at a rate of 2% per year; 6 years. **$y = 800(1.02)^x$ ≈901**
3. During a certain period of time, about 70 northern sea otters had an annual growth rate of 18%; 4 years **$y = 70(1.18)^x$ ≈136**

Write a compound interest function to model each situation. Then find the balance after the given number of years.

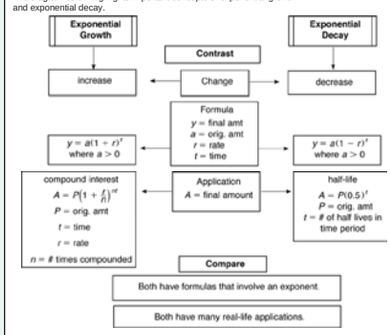
4. \$50,000 invested at a rate of 3% compounded monthly; 6 years **$y = 50{,}000(1.0025)^{12t}$ ≈\$59,847.42**
5. \$43,000 invested at a rate of 5% compounded annually; 3 years **$y = 43{,}000(1.05)^t$ ≈\$49,777.88**
6. \$65,000 invested at a rate of 6% compounded quarterly; 12 years **$y = 65{,}000(1.015)^{4t}$ ≈\$132,826.09**

Write an exponential decay function to model each situation. Then find the value of the function after the given amount of time.

7. The population of a town is 2500 and is decreasing at a rate of 3% per year; 5 years **$y = 2500(0.97)^x$ ≈2147**
8. The value of a company's equipment is \$25,000 and decreases at a rate of 15% per year; 8 years **$y = 25{,}000(0.85)^x$ ≈\$6812.26**
9. The half-life of Iodine-131 is approximately 8 days. Find the amount of Iodine-131 left from a 35 gram sample after 32 days. **2.1875 grams**

READING STRATEGIES

The diagram below highlights important concepts of exponential growth and exponential decay.



For each situation: a. identify it as exponential growth or exponential decay, and b. use a formula to calculate the answer.

1a. The bird population in an wooded area is decreasing by 3% each year from 1250. **exponential decay**

1b. Find the bird population after 6 years. **1041 birds**

2a. A town's population was 3800 in 2005 and growing at a rate of 2% every year. **exponential growth**

2b. Find the town's population in 2025. **5647 people**

3a. \$800 is invested at a rate of 4% and is compounded monthly (12 times/year). **exponential growth**

3b. Find the balance after 10 years. **\$1192.67**

RETEACH

In the exponential growth and decay formulas, y = final amount, a = original amount, r = rate of growth or decay, and t = time.

Exponential growth: $y = a(1 + r)^t$ Exponential decay: $y = a(1 - r)^t$

The population of a city is increasing at a rate of 4% each year. In 2000 there were 236,000 people in the city. Write an exponential growth function to model this situation. Then find the population in 2009.

Step 1: Identify the variables. $a = 236{,}000$ $r = 0.04$

Step 2: Substitute for a and r. $y = a(1 + r)^t$; $y = 236{,}000(1 + 0.04)^t$

The exponential growth function is $y = 236{,}000(1.04)^t$.

Growth = greater than 1.

Step 3: Substitute for t. $y = 236{,}000(1.04)^9 \approx 335{,}902$

The population will be about 335,902.

The population of a city is decreasing at a rate of 6% each year. In 2000 there were 35,000 people in the city. Write an exponential decay function to model this situation. Then find the population in 2012.

Step 1: Identify the variables. $a = 35{,}000$ $r = 0.06$

Step 2: Substitute for a and r. $y = a(1 - r)^t$; $y = 35{,}000(1 - 0.06)^t$

The exponential decay function is $y = 35{,}000(0.94)^t$.

Decay = less than 1.

Step 3: Substitute for t. $y = 35{,}000(0.94)^{12} \approx 16{,}657$

The population will be about 16,657.

Write an exponential growth function to model each situation. Then find the value of the function after the given amount of time.

1. Annual sales at a company are \$372,000 and increasing at a rate of 5% per year; 8 years **$y = 372{,}000(1 + 0.05)^8$ ≈\$549,613**
2. The population of a town is 4200 and increasing at a rate of 3% per year; 7 years **$y = 4200(1.03)^7$; ≈5165**

Write an exponential decay function to model each situation. Then find the value of the function after the given amount of time.

3. Monthly car sales for a certain type of car are \$350,000 and are decreasing at a rate of 3% per month; 6 months **$y = 350{,}000(1 - 0.03)^6$; ≈\$291,540**
4. An internet chat room has 1200 participants and is decreasing at a rate of 2% per year; 5 years **$y = 1200(0.98)^5$ ≈1085**

MULTI-STEP TEST PREP

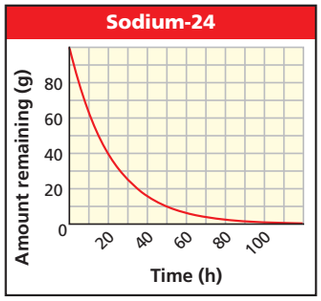
38. **a.** The annual tuition at a prestigious university was $20,000 in 2002. It generally increases at a rate of 9% each year. Write a function to describe the cost as a function of the number of years since 2002. Use 2002 as year zero when writing the function rule. $y = 20{,}000(1.09)^t$

b. What do you predict the cost of tuition will be in 2008? $33,542

c. Use a table of values to find the first year that the cost of the tuition is more than twice the cost in 2002. 2011

39. **Multi-Step** At bank A, $600 is invested with an interest rate of 5% compounded annually. At bank B, $500 is invested with an interest rate of 6% compounded quarterly. Which account will have a larger balance after 10 years? 20 years? A; B

40. **Estimation** The graph shows the decay of 100 grams of sodium-24. Use the graph to estimate the number of hours it will take the sample to decay to 10 grams. Then estimate the half-life of sodium-24. 50 h; 15 h



41. **Graphing Calculator** Use a graphing calculator to graph $y = 10(1 + r)^x$ for $r = 10\%$ and $r = 20\%$. Compare the two graphs. How does the value of r affect the graphs?

42. **Write About It** Write a real-world situation that could be modeled by $y = 400(1.08)^t$.

42. Possible answer: $400 is invested at a rate of 8% compounded annually.

43. **Write About It** Write a real-world situation that could be modeled by $y = 800(0.96)^t$. Possible answer: The population is 800 and decreasing at a rate of 4% per year.

44. **Critical Thinking** The amount of water in a container doubles every minute. After 6 minutes, the container is full. Your friend says it was half full after 3 minutes. Do you agree? Why or why not? No; possible answer: the sample doubles every minute, so the container is half full 1 min before it is full. This would be after 5 min.

TEST PREP

45. A population of 500 is decreasing by 1% per year. Which function models this situation?

Ⓐ $y = 500(0.01)^t$ Ⓑ $y = 500(0.1)^t$ Ⓒ $y = 500(0.9)^t$ Ⓓ $y = 500(0.99)^t$

46. Which function is NOT an exponential decay model?

Ⓕ $y = 5\left(\frac{1}{3}\right)^x$ Ⓖ $y = -5\left(\frac{1}{3}\right)^x$ Ⓗ $y = 5(3)^{-x}$ Ⓙ $y = 5(3^{-1})^x$

47. Stephanie wants to save $1000 for a down payment on a car that she wants to buy in 3 years. She opens a savings account that pays 5% interest compounded annually. About how much should Stephanie deposit now to have enough money for the down payment in 3 years?

Ⓐ $295 Ⓑ $333 Ⓒ $500 Ⓓ $865

48. **Short Response** In 2000, the population of a town was 1000 and was growing at a rate of 5% per year.

a. Write an exponential growth function to model this situation. $y = 1000(1.05)^t$

b. In what year is the population 1300? Show how you found your answer.

48b. about 2005; check students' work.

PROBLEM SOLVING

Write the correct answer.

1. A condo in Austin, Texas, was worth $80,000 in 1990. The value of the condo increased by an average of 3% each year. Write an exponential growth function to model this situation. Then find the value of the condominium in 2005.
$y = 80{,}000(1.03)^t$
$124,637

2. Markiya deposited $500 in a savings account. The annual interest rate is 2%, and the interest is compounded monthly. Write a compound interest function to model this situation. Then find the balance in Markiya's account after 4 years.
$y = 500\left(1 + \frac{0.02}{12}\right)^{12t}$
$541.61

3. The population of a small Midwestern town is 4500. The population is decreasing at a rate of 1.5% per year. Write an exponential decay function to model this situation. Then find the number of people in the town after 25 years.
$y = 4500(0.985)^t$
3084

4. Twelve students at a particular high school passed an advanced placement test in 2000. The number of students who passed the test increased by 16.4% each year thereafter. Find the number of students who passed the test in 2004.
22

Half-lives range from less than a second to billions of years. The table below shows the half-lives of several substances. Select the best answer.

Half-Lives	
Nitrogen-16	7 s
Technetium-99	6 h
Sulfur-35	87 days
Tritium	12.3 yr
Uranium-238	4.5 billion yrs

5. About how many grams of a 500 g sample of Technetium-99 is left after 2 days?
A 1.95 g C 31.25 g
B 7.81 g D 62.5 g

6. Which equation can be used to find how much of a 50 g sample of Nitrogen-16 is left after 7 minutes?
F $A = 50(0.5)^7$ H $A = 50(0.5)^{42}$
G $A = 50(0.5)^7$ J $A = 50(0.5)^{60}$

7. How many billions of years will it take 1000 grams of Uranium-238 to decay to just 125 grams?
A 0.125 C 9
B 3 D 13.5

8. A researcher had 37.5 g left from a 600 g sample of Sulfer-35. How many half-lives passed during that time?
F 4 H 7
G 5 J 16

9. Look at problem 8. How many days passed during that time?
A 7 C 348
B 16 D 435

CHALLENGE

Some of the parents of the students in a mathematics class invest money in accounts that pay compound interest. Their hope is to have their investments grow so that, at some time in the future, they can use the money that has accumulated to help the local Boys and Girls Club. The compound interest that they need for their financial planning is shown at right.

$A = P(1+r)^t =$
P represents the inital amount.
r represents the rate as a decimal
t represents time in years
A represents the final amount

Suppose that $10,000 is invested into an account that pays 5.65% interest compounded annually. Give answers rounded to the nearest cent.

1. Write a function that will give A as a function of t. $A = 10{,}000(1.0565)^t$
2. Complete the table below.

t	0	1	2	3	4	5	6	7
A	10,000	10,565	11,161.92	11,792.57	12,458.85	13,162.78	13,906.47	14,692.19

3. a. When will the value of the account exceed $12,000?
sometime after the end of the third year but before the end of the fourth year
b. When will the value of the account exceed $12,000 but be less than $14,000?
sometime after the end of the third year but before the end of the seventh year
4. The group of parents wants their initial deposit amount to grow to $15,000. Use guess-and-check to approximate the number of years that it will take for this to happen. Give your answer to the nearest tenth of a year. 7.4 years
5. The group wants their initial deposit amount to grow to $18,000 within 10 years. After that time, they want to make an $18,000 donation to the local club. Will they achieve their goal? Explain your response.
No; after 10 years, they will have $17,325.87, which is less than $18,000.

Use guess-and-check to approximate the time that it takes the given investment to grow to the target goal. Round answers to the nearest tenth of a year.

6. $12,500 at 4.5% compounded yearly; target goal: $18,000 8.3 years
7. $15,500 at 6.5% compounded yearly; target goal: $20,000 4.1 years
8. $5 at 5.5% compounded yearly; target goal: $18,000 153 years

MULTI-STEP TEST PREP **Exercise 38** involves writing an exponential function and using it to make a prediction. This exercise prepares students for the Multi-Step Test Prep.

TEST PREP DOCTOR In **Exercise 45,** the value of a is the same for each function. Only the value of b is different. Students who chose **A** may have forgotten to subtract the rate of decay from 1.

In **Exercise 46,** students should be looking for a value of b that is greater than 1. Students who chose **J** may have forgotten how to simplify expressions with negative exponents.

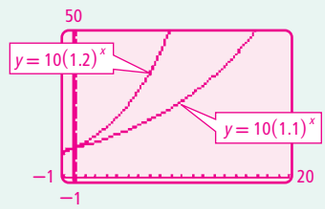
Students can solve **Exercise 47** by writing the compound interest formula, substituting the given numbers, and solving for A. After simplifying, the equation to be solved is $1000 = 1.157625A$.

Answers

38c.

Year	Tuition ($)
2002	20,000
2003	21,800
2004	23,762
2005	25,900.58
2006	28,231.63
2007	30,772.48
2008	33,542.00
2009	36,560.78
2010	39,851.25
2011	43,437.87

41.



The graph when r is 20% rises faster than when r is 10%. The greater the value of r, the faster the graph will rise.

CHALLENGE AND EXTEND

49. You invest $700 at a rate of 6% compounded quarterly. Use a graph to estimate the number of years it will take for your investment to increase to $2300. about 20 yr

50. Omar invested $500 at a rate of 4% compounded annually. How long will it take for Omar's money to double? How long would it take if the interest were 8% compounded annually? 18 yr; 9 yr

51. An 80-gram sample of a radioactive substance decayed to 10 grams after 300 minutes. Find the half-life of the substance. 100 min, or 1 h 40 min

52. Praseodymium-143 has a half-life of 2 weeks. The original measurement for the mass of a sample was lost. After 6 weeks, 15 grams of praseodymium-143 remain. How many grams was the original sample? 120 g

53. Phillip invested some money in a business 8 years ago. Since then, his investment has grown at an average rate of 1.3% compounded quarterly. Phillip's investment is now worth $250,000. How much was his original investment? Round your answer to the nearest dollar. $225,344

54. **Personal Finance** Anna has a balance of $200 that she owes on her credit card. She plans to make a $30 payment each month. There is also a 1.5% finance charge (interest) on the remaining balance each month. Copy and complete the table to answer the questions below. You may add more rows to the table as necessary.

Month	Balance ($)	Monthly Payment ($)	Remaining Balance ($)	1.5% Finance Charge ($)	New Balance ($)
1	200	30	170	2.55	172.55
2	172.55	30	142.55	2.14	144.69
3	144.69	30	114.69	1.72	116.41
4	116.41	30	86.41	1.30	87.71

a. How many months will it take Anna to pay the entire balance? 7 mo

b. By the time Anna pays the entire balance, how much total interest will she have paid? $9.01

Journal

Have students explain what each number represents in the population decay function $P = 6000(0.99)^4$.

ALTERNATIVE ASSESSMENT

Have students write and solve four different word problems, one for each of the four formulas taught in the lesson.

Power Presentations with PowerPoint®

9-3 Lesson Quiz

1. The number of employees at a certain company is 1440 and is increasing at a rate of 1.5% per year. Write an exponential growth function to model this situation. Then find the number of employees in the company after 9 years. $y = 1440(1.015)^t$; 1646

Write a compound interest function to model each situation. Then find the balance after the given number of years.

2. $12,000 invested at a rate of 6% compounded quarterly; 15 years $A = 12{,}000(1.015)^{4t}$; $29,318.64

3. $500 invested at a rate of 2.5% compounded annually; 10 years $A = 500(1.025)^t$; $640.04

4. The deer population of a game preserve is decreasing by 2% per year. The original population was 1850. Write an exponential decay function to model the situation. Then find the population after 4 years. $y = 1850(0.98)^t$; 1706

5. Iodine-131 has a half-life of about 8 days. Find the amount left from a 30-gram sample of iodine-131 after 40 days. 0.9375 g

Also available online

Answers

54.

5	87.71	30	57.71	0.87	58.58
6	58.58	30	28.58	0.43	29.01
7	29.01	29.01	0	0	0

HOUGHTON MIFFLIN HARCOURT

CHAPTER 9-3

Organizer

Pacing: Traditional 1 day
Block $\frac{1}{2}$ day

Objective: Identify and extend patterns using recursion.

Using the Extension

In working with tables representing functions, students have learned to recognize a constant *y*-difference over equal *x*-intervals as an identifier of a linear function. In this extension, students will extend their recursive pattern-recognition skills by looking for a constant ratio of successive *y*-terms as an indicator of an exponential function, and by looking for constant second- and third-degree differences to indicate quadratic and cubic functions.

COMMON CORE

CC.9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers.
CC.9-12.F.BF.2 Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.*
CC.9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input- output pairs (include reading these from a table).*

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COMMON CORE

EXTENSION

Patterns and Recursion

CC.9-12.F.IF.3 Recognize that sequences are functions, sometimes defined recursively, whose domain is a subset of the integers. *Also* **CC.9-12.F.BF.2*, CC.9-12.F.LE.2***

Objective
Identify and extend patterns using recursion.

Vocabulary
recursive pattern

In a **recursive pattern** or *recursive sequence*, each term is defined using one or more previous terms. For example, the sequence 1, 4, 7, 10, 13, ... can be defined recursively as follows: The first term is 1 and each term after the first is equal to the preceding term plus 3.

You can use recursive techniques to identify patterns. The table summarizes the characteristics of four types of patterns.

Using Recursive Techniques to Identify Patterns

Type of Pattern	Characteristics
Linear	First differences are constant.
Quadratic	Second differences are constant.
Cubic	Third differences are constant.
Exponential	Ratios between successive terms are constant.

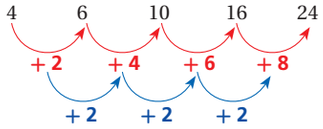
EXAMPLE 1 Identifying and Extending a Pattern

Identify the type of pattern. Then find the next three numbers in the pattern.

Helpful Hint
You may need to use trial and error when identifying a pattern. If first, second, and third differences are not constant, check for constant ratios.

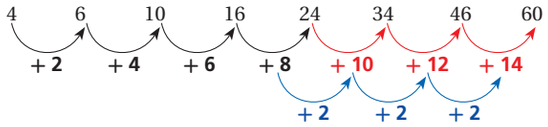
A **4, 6, 10, 16, 24, ...**

Find first, second, and, if necessary, third differences.



Second differences are constant, so the pattern is quadratic.

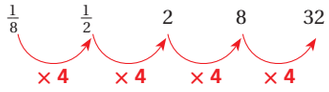
Extend the pattern by continuing the sequence of first and second differences.



The next three numbers in the pattern are 34, 46, and 60.

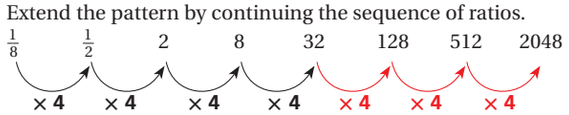
B $\frac{1}{8}, \frac{1}{2}$, **2, 8, 32**

Find the ratio between successive terms.



Ratios between terms are constant, so the pattern is exponential.

Extend the pattern by continuing the sequence of ratios.



The next three numbers in the pattern are 128, 512, and 2048.

1 Introduce

Motivate

Write the sequence 5, 10, 15, 20, 25, ... on the board. Ask students to describe the pattern. In particular, ask them to give a rule that they could use to find the next few numbers in the pattern. Possible answer: Add 5 to each term to get the next term. Explain that there are different ways to describe the pattern and give a rule for the pattern. Point out that the method in which each term is found by doing something to a previous term is called *recursion*.

2 Teach

Guided Instruction

Remind students that they have already worked with first differences, second differences, and ratios between successive terms. Tell them that this lesson pulls these ideas together for identifying and describing both patterns and functions. When you discuss **Example 2,** be sure to emphasize that the method of looking for constant differences or ratios in the *y*-values only works when the *x*-values are evenly spaced.

CHECK IT OUT! **Identify the type of pattern. Then find the next three numbers in the pattern.**

1a. 56, 47, 38, 29, 20, ...
linear; 11, 2, –7

1b. 1, 8, 27, 64, 125, ...
cubic; 216, 343, 512

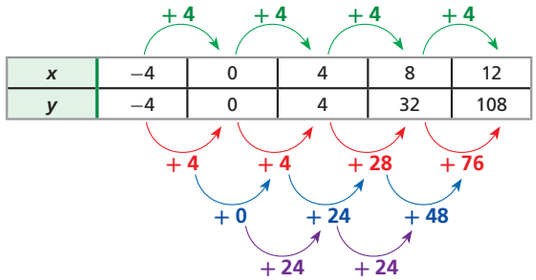
You can use a similar process to determine whether a function is linear, quadratic, cubic, or exponential. Note that before comparing *y*-values, you must first make sure there is a constant change in the corresponding *x*-values.

Using Recursive Techniques to Identify Functions	
Type of Function	**Characteristics (Given a Constant Change in *x*-values)**
Linear	First differences of *y*-values are constant.
Quadratic	Second differences of *y*-values are constant.
Cubic	Third differences of *y*-values are constant.
Exponential	Ratios between successive *y*-values are constant.

EXAMPLE 2 Identifying a Function

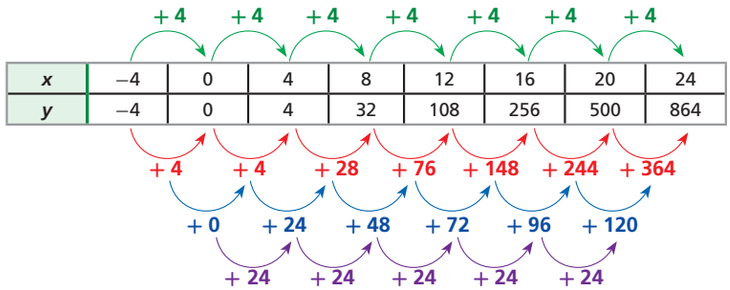
The ordered pairs {(–4, –4), (0, 0), (4, 4), (8, 32), (12, 108)} satisfy a function. Determine whether the function is linear, quadratic, cubic, or exponential. Then find three additional ordered pairs that satisfy the function.

Make a table. Check for a constant change in the *x*-values. Then find first, second, and third differences of *y*-values.



There is a constant change in the *x*-values. Third differences are constant. The function is a cubic function.

To find additional ordered pairs, extend the pattern by working backward from the constant third differences.



Three additional ordered pairs that satisfy this function are (16, 256), (20, 500), and (24, 864).

Helpful Hint

In Example 2, the constant third differences are 24. To extend the pattern, first find each second difference by adding 24 to the previous second difference. Then find each first difference by adding the second difference below to the previous first difference.

Additional Examples

Example 1

Identify the type of pattern. Then find the next three numbers in the pattern.

A. 16, 54, 128, 250, 432, ...
cubic; 686, 1024, 1458

B. 800, 400, 200, 100, 50, ...
exponential; 25, 12.5, 6.25

Example 2

Determine whether each function is linear, quadratic, or exponential.

A.

x	0	2	4	6	8
y	2	5	17	38	68

quadratic

B.

x	–3	1	5	9	13
y	2	4	8	16	32

exponential

MATHEMATICAL PRACTICES **Questioning Strategies**

EXAMPLE 1

- To find first differences, in which order should you subtract consecutive terms?
- Is it possible for first differences to be positive? negative? zero?

EXAMPLE 2

- What is the first thing you should do when trying to determine what type of function is shown in a table?
- How is identifying a function from a table similar to identifying a pattern from a sequence of numbers? How is it different?

3 Close

Summarize

Ask students to summarize in their own words how they can recognize a linear, quadratic, cubic, or exponential pattern or function.

Possible answer:
Linear: constant first differences
Quadratic: constant second differences
Cubic: constant third differences
Exponential: constant ratios

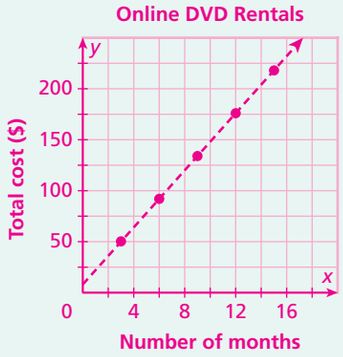
Teaching Tip **Multiple Representations**

A quick demonstration can connect the ideas of this lesson to what students know about equations. Starting with a quadratic equation, such as $y = x^2 + 1$, generate a table of values with evenly spaced *x*-values. Then check that the second differences of the *y*-values are constant.

Answers

15a. Linear; for a constant change in the number of months, first differences of costs are constant.

b.



c. The data lie on a line; the function is linear.

d. $260

2b. exp.; $\left(11, \frac{1}{486}\right), \left(13, \frac{1}{1458}\right), \left(15, \frac{1}{4374}\right)$

CHECK IT OUT!

Several ordered pairs that satisfy a function are given. Determine whether the function is linear, quadratic, cubic, or exponential. Then find three additional ordered pairs that satisfy the function.

2a. {(0, 1), (1, 3), (2, 9), (3, 19), (4, 33)} quad.; (5, 51), (6, 73), (7, 99)

2b. $\left\{\left(1, \frac{1}{2}\right), \left(3, \frac{1}{6}\right), \left(5, \frac{1}{18}\right), \left(7, \frac{1}{54}\right), \left(9, \frac{1}{162}\right)\right\}$

EXTENSION Exercises

Identify the type of pattern. Then find the next three numbers in the pattern.

1. 25, 28, 31, 34, 37, ... linear; 40, 43, 46
2. 20, 45, 80, 125, 180, ... quad.; 245, 320, 405
3. 128, 64, 32, 16, 8, ... exp.; 4, 2, 1
4. 4, 32, 108, 256, 500, ... cubic; 864, 1372, 2048
5. $\frac{1}{2}, \frac{3}{4}, 1, 1\frac{1}{4}, 1\frac{1}{2}, \ldots$ linear; $1\frac{3}{4}, 2, 2\frac{1}{4}$
6. 0.3, 0.03, 0.003, 0.0003, 0.00003, ... exp.; 0.000003, 0.0000003, 0.00000003
7. 127, 66, 29, 10, 3, ... cubic; 2, 1, −6
8. 2, 8, 18, 32, 50, ... quad.; 72, 98, 128

Several ordered pairs that satisfy a function are given. Determine whether the function is linear, quadratic, cubic, or exponential. Then find three additional ordered pairs that satisfy the function.

9. {(3, 1), (5, −3), (7, −7), (9, −11), (11, −15)} linear; (13, −19), (15, −23), (17, −27)
10. {(−1, −2), (2, 7), (5, 124), (8, 511), (11, 1330)} cubic; (14, 2743), (17, 4912), (20, 7999)
11. $\left\{\left(2, \frac{1}{4}\right), \left(3, \frac{1}{8}\right), \left(4, \frac{1}{16}\right), \left(5, \frac{1}{32}\right), \left(6, \frac{1}{64}\right)\right\}$ exp.; $\left(7, \frac{1}{128}\right), \left(8, \frac{1}{256}\right), \left(9, \frac{1}{512}\right)$
12. {(−3, −7), (0, 2), (3, −7), (6, −34), (9, −79)} quad.; (12, −142), (15, −223), (18, −322)
13. {(0, 600), (10, 480), (20, 384), (30, 307.2), (40, 245.76)} exp.; (50, 196.608), (60, 157.2864), (70, 125.82912)
14. {(−8, 2), (−5, 7), (−2, 12), (1, 17), (4, 22)} linear; (7, 27), (10, 32), (13, 37)

15. **Entertainment** The table shows the cost of using an online DVD rental service for different numbers of months.

Online DVD Rentals	
Months	Cost ($)
3	50
6	92
9	134
12	176
15	218

   a. Determine whether the function that models the data is linear, quadratic, cubic, or exponential. Explain.
   b. Graph the data in the table.
   c. What do you notice about your graph? Why does this make sense?
   d. Predict the cost of the service for 18 months.

16. A student claimed that the function shown in the table is a quadratic function. Do you agree or disagree? Explain.

x	3	7	10	14	17
y	2	6	12	20	30

+4, +6, +8, +10; +2, +2, +2

Disagree; the table does not show a constant change in *x*-values.

17. Business The table shows the annual sales for a small company.

Annual Sales	
Year	Sales ($)
2006	513,000
2007	516,000
2008	521,000
2009	528,000
2010	537,000

a. Determine whether the function that models the data is linear, quadratic, cubic, or exponential. Explain.

b. Suppose sales continue to grow according to the pattern in the table. Predict the annual sales for 2011, 2012, and 2013.

c. If the pattern continues, in what year will annual sales be $17,000 greater than the previous year's sales?

18. Critical Thinking Use the table for the following problems.

x	0	1	2	3	4
y	3	6			

a. Copy and complete the table so that the function is a linear function.

b. Copy and complete the table so that the function is a quadratic function.

c. Copy and complete the table so that the function is an exponential function.

d. For which of these three types of functions is there more than one correct way to complete the table? Explain.

Use the description to write the first five terms in each numerical pattern.

19. The first term is 8. Each following term is 11 less than the term before it. **8, −3, −14, −25, −36**

20. The first term is 1000. Each following term is 40% of the term before it. **1000, 400, 160, 64, 25.6**

21. The first two terms are 1 and 2. Each following term is the sum of the two terms before it. **1, 2, 3, 5, 8**

Make a table for a function that has the given characteristics. Include at least five ordered pairs.

22. The function is linear. The first differences are −3.

23. The function is quadratic. The second differences are 6.

24. The function is cubic. The third differences are 1.

A *recursive formula* for a sequence shows how to find the value of a term from one or more terms that come before it. For example, the recursive formula $a_n = a_{n-1} + 3$ tells you that each term is equal to the preceding term plus 3. Given that $a_1 = 5$, you can use the formula to generate the sequence 5, 8, 11, 14, ...

Write the first four terms of each sequence.

25. $a_n = a_{n-1} + 2$; $a_1 = 12$ **12, 14, 16, 18**

26. $a_n = a_{n-1} - 7$; $a_1 = 16$ **16, 9, 2, −5**

27. $a_n = 2a_{n-1}$; $a_1 = 4$ **4, 8, 16, 32**

28. $a_n = 0.6a_{n-1}$; $a_1 = 100$ **100, 60, 36, 21.6**

29. $a_n = 5a_{n-1} - 2$; $a_1 = 0$ **0, −2, −12, −62**

30. $a_n = (a_{n-1})^2$; $a_1 = -2$ **−2, 4, 16, 256**

31. A *recursive function* defines a function for whole numbers by referring to the value of the function at previous whole numbers. Consider the recursive function $f(n) = f(n-1) + 5$ with $f(0) = 1$.

a. According to the formula, $f(1) = f(0) + 5$. What is the value of $f(1)$? **6**

b. Use the formula to find $f(2)$, $f(3)$, $f(4)$, and $f(5)$. **11, 16, 21, 26**

c. Graph $f(n)$ by plotting points at $x = 0$, $x = 1$, $x = 2$, $x = 3$, $x = 4$, and $x = 5$.

d. What do you notice about your graph? What does this tell you about $f(n)$?

Answers

17a. Quadratic; for a constant change in years, second differences of sales are constant.

b. $548,000; $561,000; $576,000

c. 2014

18a.

x	0	1	2	3	4
y	3	6	9	12	15

b. Possible answer:

x	0	1	2	3	4
y	3	6	11	18	27

c.

x	0	1	2	3	4
y	3	6	12	24	48

d. Quadratic; you can choose any value for the constant second difference.

22-24. Possible answers given.

22.

x	0	1	2	3	4
y	4	1	−2	−5	−8

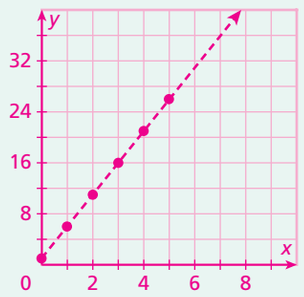
23.

x	−2	−1	0	1	2
y	1	5	15	31	53

24.

x	1	2	3	4	5
y	7	4	6	14	29

31c.



d. The points lie on a line; the function is linear.

HOUGHTON MIFFLIN HARCOURT

COMMON CORE

9-4 Linear, Quadratic, and Exponential Models

CC.9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.* *Also* **CC.9-12.F.LE.2*, CC.9-12.A.CED.2*, CC.9-12.F.IF.4*, CC.9-12.F.IF.7*, CC.9-12.F.BF.1***

Objectives
Compare linear, quadratic, and exponential models.

Given a set of data, decide which type of function models the data and write an equation to describe the function.

Why learn this?
Different situations in sports can be described by linear, quadratic, or exponential models.

Look at the tables and graphs below. The data show three ways you have learned that variable quantities can be related. The relationships shown are linear, quadratic, and exponential.

Linear

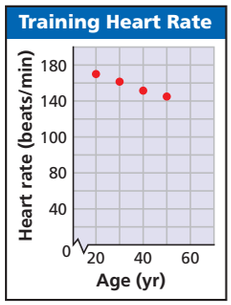
Training Heart Rate	
Age (yr)	Beats/min
20	170
30	161.5
40	153
50	144.5

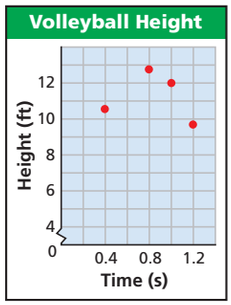
Quadratic

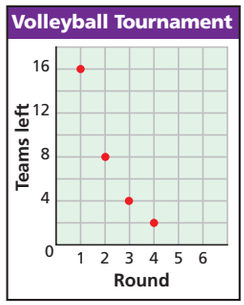
Volleyball Height	
Time (s)	Height (ft)
0.4	10.44
0.8	12.76
1	12
1.2	9.96

Exponential

Volleyball Tournament	
Round	Teams Left
1	16
2	8
3	4
4	2







In the real world, people often gather data and then must decide what kind of relationship (if any) they think best describes their data.

EXAMPLE 1 **Graphing Data to Choose a Model**

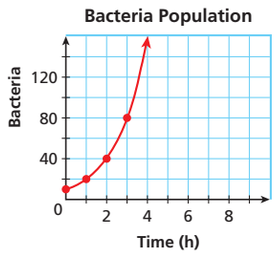
Graph each data set. Which kind of model best describes the data?

A

Time (h)	0	1	2	3
Bacteria	10	20	40	80

Plot the data points and connect them.

The data appear to be exponential.



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9-4 Organizer

Pacing: Traditional 2 days
Block 1 day

Objectives: Compare linear, quadratic, and exponential models.

Given a set of data, decide which type of function models the data and write an equation to describe the function.

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Warm Up

1. Find the slope and the *y*-intercept of the line that passes through (4, 20) and (20, 24). $\frac{1}{4}$; 19

The population of a town is decreasing at a rate of 1.8% per year. In 1990, there were 4600 people.

2. Write an exponential decay function to model this situation. $y = 4600(0.982)^t$
3. Find the population in 2010. 3199

Also available online

Math Humor

Teacher: Why are you so dressed up?

Student: You said we'd be modeling today.

COMMON CORE

CC.9-12.F.LE.1 Distinguish between situations that can be modeled with linear functions and with exponential functions.*

CC.9-12.F.LE.2 Construct linear and exponential functions, including arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs (include reading these from a table).*

CC.9-12.A.CED.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.*

Learn It Online
State Resources Online

1 Introduce

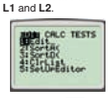
EXPLORATION

Linear, Quadratic, and Exponential Models

You will need a graphing calculator for this Exploration.

1. To enter the data from the table below into your calculator, press STAT and select **1:Edit...** . Then enter the values from the table in lists **L1** and **L2**.

x	y
0	5
1	2
2	1
3	2
4	5



2. To plot the data points, press Y= and select **Plot 1** at the top of the screen. Then press GRAPH.
3. Based on the shape of the graph, what type of function do you think models the data? Why?
4. Repeat the process for the data in this table.

x	y
1	7.2
2	5.8
3	4.6
4	3.7
5	2.9

THINK AND DISCUSS

5. **Explain** what type of function you think models the data in Problem 4.

Motivate

Display the following table and have students list information they remember about each type of function.

Linear	Quadratic	Exponential
Possible answers: graph is a line, has a slope	Possible answers: graph is a parabola, opens up or down	Possible answers: graph is a curve, variable is in exponent

Explorations and answers are provided in your online resources.

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Additional Examples

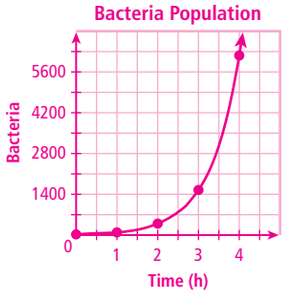
Example 1

Graph each data set. Which kind of model best describes the data?

A.

Time (h)	Bacteria
0	24
1	96
2	384
3	1536
4	6144

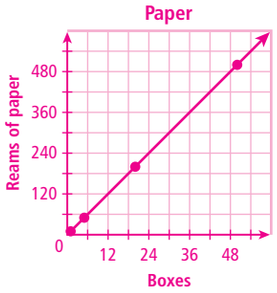
exponential



B.

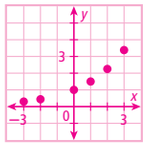
Boxes	Reams of Paper
1	10
5	50
20	200
50	500

linear



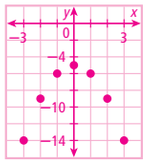
Also available online

1a.



exponential

1b.



quadratic

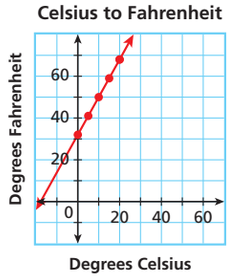
Graph each data set. Which kind of model best describes the data?

B

°C	0	5	10	15	20
°F	32	41	50	59	68

Plot the data points and connect them.

The data appear to be linear.



CHECK IT OUT! **Graph each data set. Which kind of model best describes the data?**

1a. $\{(-3, 0.30), (-2, 0.44), (0, 1), (1, 1.5), (2, 2.25), (3, 3.38)\}$

1b. $\{(-3, -14), (-2, -9), (-1, -6), (0, -5), (1, -6), (2, -9), (3, -14)\}$

Another way to decide which kind of relationship (if any) best describes a data set is to use patterns.

EXAMPLE 2 Using Patterns to Choose a Model

Look for a pattern in each data set to determine which kind of model best describes the data.

A

Height of Bridge Suspension Cables

Cable's Distance from Tower (ft)	Cable's Height (ft)
0	400
100	256
200	144
300	64

+ 100, + 100, + 100; − 144, − 112, − 80; + 32, + 32

For every constant change in distance of +100 feet, there is a constant second difference of +32.

The data appear to be quadratic.

B

Value of a Car

Car's Age (yr)	Value ($)
0	20,000
1	17,000
2	14,450
3	12,282.50

+ 1, + 1, + 1; × 0.85, × 0.85, × 0.85

For every constant change in age of +1 year, there is a constant ratio of 0.85.

The data appear to be exponential.

Remember!

When the independent variable changes by a constant amount,
- linear functions have constant first differences.
- quadratic functions have constant second differences.
- exponential functions have a constant ratio.

CHECK IT OUT! **2.** Look for a pattern in the data set $\{(-2, 10), (-1, 1), (0, -2), (1, 1), (2, 10)\}$ to determine which kind of model best describes the data. **quadratic**

MATHEMATICAL PRACTICES Questioning Strategies

EXAMPLE 1
- What does the graph of a linear function look like?
- How are the graphs of quadratic and exponential functions similar? How are they different?

2 Teach

Guided Instruction

Show how to choose a model graphically and then by looking at patterns. **Check It Out Problem 3** involves finding the slope and *y*-intercept of a linear function. These skills may need to be reviewed. This lesson does not include finding a function to model quadratic data; that is beyond the scope of this book. Interested or advanced students can find the function for **Example 2A** on a graphing calculator by entering the data into two lists and using the **QuadReg** feature.

Teaching Tip **Inclusion** When graphing, students might choose the wrong model if their graph is sloppy or if they do not graph all the points. For example, points that appear to be parabolic may actually be exponential. Tell students that when in doubt, they should look for patterns.

Teaching Tip **Critical Thinking** Remind students that $y \neq 0$ in an exponential function. If the data set includes $y = 0$, then the function must be linear or quadratic.

After deciding which model best fits the data, you can write a function. Recall the general forms of linear, quadratic, and exponential functions.

Know it! Note

General Forms of Functions

LINEAR	QUADRATIC	EXPONENTIAL
$y = mx + b$	$y = ax^2 + bx + c$	$y = ab^x$

EXAMPLE 3 *Problem-Solving Application*

Use the data in the table to describe how the ladybug population is changing. Then write a function that models the data. Use your function to predict the ladybug population after one year.

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them.

Ladybug Population	
Time (mo)	Ladybugs
0	10
1	30
2	90
3	270

1 Understand the Problem

The **answer** will have three parts—a description, a function, and a prediction.

2 Make a Plan

Determine whether the data is linear, quadratic, or exponential. Use the general form to write a function. Then use the function to find the population after one year.

3 Solve

Step 1 Describe the situation in words.

Ladybug Population		
	Time (mo)	Ladybugs
+1	0	10 ×3
+1	1	30 ×3
+1	2	90 ×3
+1	3	270

Each month, the ladybug population is multiplied by 3. In other words, the population triples each month.

Step 2 Write the function.

There is a constant ratio of 3. The data appear to be exponential.

$y = ab^x$ *Write the general form of an exponential function.*

$y = a(3)^x$ *Substitute the constant ratio, 3, for b.*

$10 = a(3)^0$ *Choose an ordered pair from the table, such as (0, 10). Substitute for x and y.*

$10 = a(1)$ *Simplify. $3^0 = 1$*

$10 = a$ *The value of a is 10.*

$y = 10(3)^x$ *Substitute 10 for a in $y = a(3)^x$.*

Helpful Hint

You can choose any given ordered pair to substitute for x and y. However, it is often easiest to choose an ordered pair that contains 0.

Christine Lee Zilka/Getty Images

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Additional Examples

Example 2

Look for a pattern in each data set to determine which kind of model best describes the data.

A.

Height of Golf Ball	
Time (s)	Height (ft)
0	4
1	68
2	100
3	100
4	68

quadratic

B.

Money in CD	
Time (yr)	Amount ($)
0	1000.00
1	1169.86
2	1368.57
3	1601.04

exponential

Example 3

Use the data in the table to describe how the number of people changes. Then write a function that models the data. Use your function to predict the number of people who received the e-mail after one week.

E-mail Forwarding	
Time (days)	Number of People Who Received the E-mail
0	8
1	56
2	392
3	2744

increasing by a factor of 7 each day; $y = 8(7)^x$; 6,588,344

Also available online

MATHEMATICAL PRACTICES **Questioning Strategies**

EXAMPLE 2

- How can you use the pattern in the table to predict the contents of the next row?

EXAMPLE 3

- How can you use the function to predict other data values?
- How accurate is this prediction?

Reaching All Learners *Through Cooperative Learning*

Break students into small groups. Assign each group one of the following kinds of models: linear, quadratic, or exponential. Tell students to prepare a 5-minute presentation that covers:

- characteristics of their model
- how to recognize when their model best describes a set of data
- examples of real-life situations that tend to fit their model

Teaching Tip **Visual** Students may get a better understanding of exponential functions if they attempt to draw a tree diagram of the situation. For the data in **Example 3,** they should start by drawing 10 circles for 10 bugs and then draw 3 circles for each of those 10 circles, and so on.

Step 3 Predict the ladybug population after one year.

$y = 10(3)^x$ *Write the function.*

$= 10(3)^{12}$ *Substitute 12 for x (1 year = 12 mo).*

$= 5{,}314{,}410$ *Use a calculator.*

There will be 5,314,410 ladybugs after one year.

4 Look Back

You chose the ordered pair $(0, 10)$ to write the function. Check that every other ordered pair in the table satisfies your function.

$y = 10(3)^x$	
30	$10(3)^1$
30	$10(3)$
30	30 ✓

$y = 10(3)^x$	
90	$10(3)^2$
90	$10(9)$
90	90 ✓

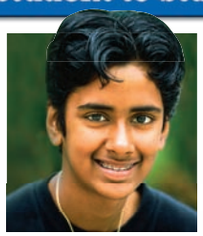
$y = 10(3)^x$	
270	$10(3)^3$
270	$10(27)$
270	270 ✓

CHECK IT OUT! **3.** Use the data in the table to describe how the oven temperature is changing. Then write a function that models the data. Use your function to predict the temperature after 1 hour.

The oven temperature decreases by 50 °F every 10 min; $y = -5x + 375$; 75 °F

Oven Temperature				
Time (min)	0	10	20	30
Temperature (°F)	375	325	275	225

Student to Student — *Checking Units*



Michael Gambhir
Warren High School

I used to get a lot of answers wrong because of the units. If a question asked for the value of something after 1 year, I would always just substitute 1 into the function.

I finally figured out that you have to check what x is. If x represents months and you're trying to find the value after 1 year, then you have to substitute 12, not 1, because there are 12 months in a year.

MATHEMATICAL PRACTICES

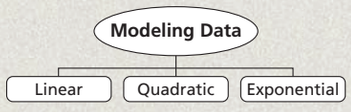
THINK AND DISCUSS

1. Do you think that every data set will be able to be modeled by a linear, quadratic, or exponential function? Why or why not?

2. In Example 3, is it certain that there will be 5,314,410 ladybugs after one year? Explain.

Know it! Note

3. GET ORGANIZED Copy and complete the graphic organizer. In each box, list some characteristics and sketch a graph of each type of model.



age fotostock

3 Close

Summarize

Ask students which model best fits each of the following situations:

- With a constant change in x, the differences between the first differences are constant. quadratic
- With a constant change in x, there is a constant ratio in the y-values. exponential
- The graph forms a parabola. quadratic
- With a constant change in x, the second differences are zero. linear

ONGOING ASSESSMENT and INTERVENTION

***Diagnose Before* the Lesson**
Warm Up, TE

***Monitor During* the Lesson**
Check It Out! Exercises, SE
Questioning Strategies, TE

***Assess After* the Lesson**
Lesson Quiz, TE
Alternative Assessment, TE

Answers to *Think and Discuss*

Possible answers:

1. No; most real-world data probably will not fit exactly into one of these patterns.
2. No; this is just a prediction based on the assumption that the observed trends will continue, which they may or may not do.
3. See Additional Answers.

9-4 Exercises

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GUIDED PRACTICE

SEE EXAMPLE 1

Graph each data set. Which kind of model best describes the data?

1. $\{(-1, 4), (-2, 0.8), (0, 20), (1, 100), (-3, 0.16)\}$ exponential
2. $\{(0, 3), (1, 9), (2, 11), (3, 9), (4, 3)\}$ quadratic
3. $\{(2, -7), (-2, -9), (0, -8), (4, -6), (6, -5)\}$ linear

SEE EXAMPLE 2

Look for a pattern in each data set to determine which kind of model best describes the data.

4. $\{(-2, 1), (-1, 2.5), (0, 3), (1, 2.5), (2, 1)\}$ quadratic
5. $\{(-2, 0.75), (-1, 1.5), (0, 3), (1, 6), (2, 12)\}$ exponential
6. $\{(-2, 2), (-1, 4), (0, 6), (1, 8), (2, 10)\}$ linear

SEE EXAMPLE 3

7. **Consumer Economics** Use the data in the table to describe the cost of grapes. Then write a function that models the data. Use your function to predict the cost of 6 pounds of grapes.

Total Cost of Grapes				
Amount (lb)	1	2	3	4
Cost ($)	1.79	3.58	5.37	7.16

Grapes cost \$1.79/lb; $y = 1.79x$; \$10.74

PRACTICE AND PROBLEM SOLVING

Independent Practice

For Exercises	See Example
8–10	1
11–13	2
14	3

Extra Practice
See Extra Practice for more Skills Practice and Applications Practice exercises.

Graph each data set. Which kind of model best describes the data?

8. $\{(-3, -5), (-2, -8), (-1, -9), (0, -8), (1, -5), (2, 0), (3, 7)\}$ quadratic
9. $\{(-3, -1), (-2, 0), (-1, 1), (0, 2), (1, 3), (2, 4), (3, 5)\}$ linear
10. $\{(0, 0.1), (2, 0.9), (3, 2.7), (4, 8.1)\}$ exponential

Look for a pattern in each data set to determine which kind of model best describes the data.

11. $\{(-2, 5), (-1, 4), (0, 3), (1, 2), (2, 1)\}$ linear
12. $\{(-2, 12), (-1, 15), (0, 16), (1, 15), (2, 12)\}$ quadratic
13. $\{(-2, 8), (-1, 4), (0, 2), (1, 1), (2, 0.5)\}$ exponential

14. **Business** Use the data in the table to describe how the company's sales are changing. Then write a function that models the data. Use your function to predict the amount of sales after 10 years.

14. The company's sales are increasing by 20% each year; $y = 25{,}000(1.2)^x$; \$154,793.41.

Company Sales				
Year	0	1	2	3
Sales ($)	25,000	30,000	36,000	43,200

15. **Multi-Step** Jay's hair grows about 6 inches each year. Write a function that describes the length ℓ in inches that Jay's hair will grow for each year k. Which kind of model best describes the function? $\ell = 6k$; linear

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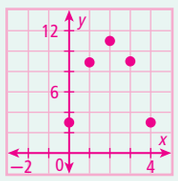

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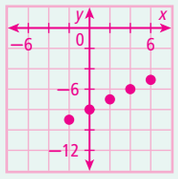

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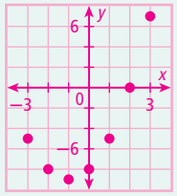

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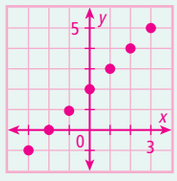

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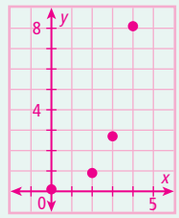


9-4 Exercises

Assignment Guide

Assign *Guided Practice* exercises as necessary.

If you finished Examples **1–2**
Basic 8–13, 16–18, 21, 22
Average 8–13, 16–18, 21, 22
Advanced 8–13, 16–18, 21, 22

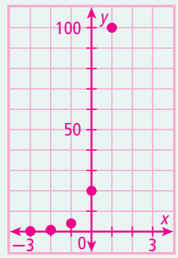
If you finished Examples **1–3**
Basic 8–21, 24, 26–29
Average 8–19, 23–29
Advanced 8–19, 23, 25–31

Homework Quick Check
Quickly check key concepts.
Exercises: 8, 10, 12, 14, 18, 20

Teaching Tip

Inclusion In **Exercise 10,** there is not a constant change in x. Have students focus on the last three points and then use that to find y when x is 1. Students should then check whether the pattern works for all four points.

Answers

1.


MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them. Exercises 7, 14–18, 26, 30–31

Construct viable arguments and critique the reasoning of others. Exercises 23–25

Look for and express regularity in repeated reasoning. Exercises 4–6, 11–13

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MULTI-STEP TEST PREP **Exercise 26** involves writing a function rule to describe data. This exercise prepares students for the Multi-Step Test Prep.

Answers

26a. college 1: linear; college 2: exponential

b. college 1: $y = 200x + 2000$; college 2: $y = 2000(1.1)^x$

c. Both have the same tuition ($2000) in 2000.

d. For college 1, $200 is added each year, so 2000 + 200 = 2200. For college 2, 10% is added each year, so 2000 + (0.1)(2000) = 2200.

Tell which kind of model best describes each situation.

16. The height of a plant at weekly intervals over the last 6 weeks was 1 inches, 1.5 inches, 2 inches, 2.5 inches, 3 inches., and 3.5 inches. **linear**

17. The number of games a baseball player played in the last four years was 162, 162, 162, and 162. **linear**

18. The height of a ball in a certain time interval was recorded as 30.64 feet, 30.96 feet, 31 feet, 30.96 feet, and 30.64 feet. **quadratic**

Write a function to model each set of data.

19. $y = 0.2(4)^x$

20. $y = -\frac{1}{2}x + 4$

21. linear

22. quadratic

23. Possible answer: (0, 3), (1, 6), (2, 12), (3, 24); for a constant change in x of +1, there is a common ratio of 2.

19.

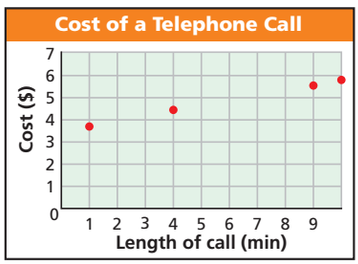
x	−1	0	1	2	4
y	0.05	0.2	0.8	3.2	51.2

20.

x	−2	0	2	4	8
y	5	4	3	2	0

Tell which kind of model best describes each graph.

21.



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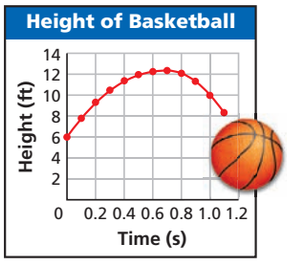
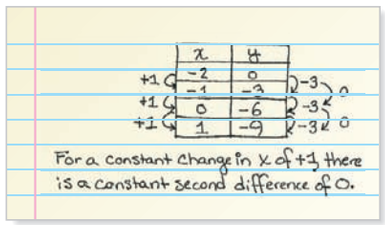


23. Write About It Write a set of data that you could model with an exponential function. Explain why the exponential model would work.

24. Possible answer: The first differences are constant, so there is no need to check the second differences. A linear function would best model the data.

24. ERROR ANALYSIS A student concluded that the data set would best be modeled by a quadratic function. Explain the student's error.



25. Critical Thinking Sometimes the graphs of quadratic data and exponential data can look very similar. Describe how you can tell them apart.

Possible answer: Make a table of ordered pairs and see whether the y-values show a pattern of constant second differences or constant ratios.

MULTI-STEP TEST PREP

26. a. Examine the two models that represent annual tuition for two colleges. Describe each model as linear, quadratic, or exponential.

b. Write a function rule for each model.

c. Both models have the same values for 2004. What does this mean?

d. Why do both models have the same value for year 1?

Years After 2004	Tuition at College 1 ($)	Tuition at College 2 ($)
0	2000.00	2000.00
1	2200.00	2200.00
2	2400.00	2420.00
3	2600.00	2662.00
4	2800.00	2928.20

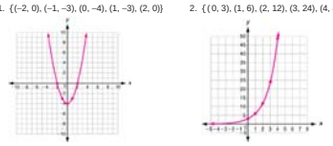
PRACTICE A

PRACTICE C

PRACTICE B

Graph each data set. Which kind of model best describes the data?

1. {(−2, 0), (−1, −3), (0, −4), (1, −3), (2, 0)}
2. {(0, 3), (1, 6), (2, 12), (3, 24), (4, 48)}



quadratic — exponential

Look for a pattern in each data set to determine which kind of model best describes the data.

3. {(−5, 9), (−4, 0), (−3, −7), (−2, −12)} — quadratic
4. {(−2, 9), (−1, 13), (0, 17), (1, 21)} — linear
5. {(1, 4), (2, 6), (3, 9), (4, 13.5)} — exponential
6. {(0, 4), (2, 12), (4, 36), (6, 76)} — quadratic
7. $\left\{(1,17), \left(3, 8\frac{1}{2}\right), \left(5, 4\frac{1}{2}\right), \left(7, 2\frac{1}{8}\right)\right\}$ — exponential

8. Use the data in the table to describe how the restaurant's sales are changing. Then write a function that models the data. Use your function to predict the amount of sales after 8 years.

Restaurant				
Year	0	1	2	3
Sales ($)	20,000	19,000	18,050	17,147.50

The sales decrease by 5% each month; $y = 20{,}000(0.95)^x$; ≈$13,268.41

9. Use the data in the table to describe how the clothing store's sales are changing. Then write a function that models the data. Use your function to predict the amount of sales after 10 years.

Clothing Store				
Year	0	1	2	3
Sales ($)	15,000	15,750	16,500	17,250

The sales increase by $750 every year; $y = 750x + 15{,}000$; $22,500

READING STRATEGIES

Use the table below to help you learn the characteristics of each data model and how to tell them apart.

Model	Linear	Quadratic	Exponential
Pattern (y-values)	constant first differences	constant second differences	constant ratios
Graph			
Equation	$y = mx + b$	$y = ax^2 + bx + c$	$y = ab^x$

Identify each of the following as *linear*, *quadratic*, or *exponential*.

1. $y = 6(2)^x$ — exponential
2. $y = 4x + 6$ — linear
3. $y = 2x^2 + 5x + 3$ — quadratic

4. quadratic
5. exponential
6. linear

7.

x	−6	−7	−8	−9	−10
y	17	20	23	26	29

linear

8.

x	1	3	5	7	9
y	4	12	28	52	84

quadratic

9.

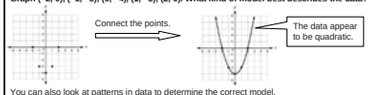
x	5	4	3	2	1
y	128	64	32	16	8

exponential

RETEACH

Graph to decide whether data is best modeled by a linear, quadratic or exponential function.

Graph (−2, 0), (−1, −3), (0, −4), (1, −3), (2, 0). What kind of model best describes the data?



You can also look at patterns in data to determine the correct model.

Linear functions have constant 1st differences.

x	y	
2	5	−3
4	2	−3
6	−1	−3
8	−4	

Quadratic functions have constant 2nd differences.

x	y		
1	−8	+3	
2	−5	+5	+2
3	0	+7	+2
4	7		

Exponential functions have a constant ratio.

x	y	
0	−2	×4
1	−8	×4
2	−32	×4
3	−128	

Graph each data set. Which kind of model best describes the data?

1. (−2, −4), (−1, −2), (0, 0), (1, 2), (2, 4) — linear
2. (−1, 4), (0, 2), (1, 1), $\left(2, \frac{1}{2}\right)$, $\left(3, \frac{1}{4}\right)$ — exponential

Look for a pattern in each data set to determine which kind of model best describes the data.

3. exponential

x	y
0	6
1	12
2	24
3	48

4. quadratic

x	y
0	10
1	18
2	28
3	40

5. linear

x	y
3	4
6	−2
9	−8
12	−14

TEST PREP

27. Which function best models the data: $\{(-4, -2), (-2, -1), (0, 0), (2, 1), (4, 2)\}$?

Ⓐ $y = \left(\frac{1}{2}\right)^x$ Ⓑ $y = \frac{1}{2}x^2$ Ⓒ $y = \frac{1}{2}x$ Ⓓ $y = \left(\frac{1}{2}x\right)^2$

28. A city's population is increasing at a rate of 2% per year. Which type of model describes this situation?

Ⓕ Exponential Ⓖ Quadratic Ⓗ Linear Ⓙ None of these

29. Which data set is best modeled by a linear function?

Ⓐ $\{(-2, 0), (-1, 2), (0, -4), (1, -1), (2, 2)\}$

Ⓑ $\{(-2, 2), (-1, 4), (0, 6), (1, 16), (2, 32)\}$

Ⓒ $\{(-2, 2), (-1, 4), (0, 6), (1, 8), (2, 10)\}$

Ⓓ $\{(-2, 0), (-1, 5), (0, 7), (1, 5), (2, 0)\}$

CHALLENGE AND EXTEND

30a.

Year	Value ($)
0	18,000
1	15,120
2	12,700.80
3	10,668.67
4	8961.68

30. **Finance** An accountant estimates that a certain new automobile worth $18,000 will lose value at a rate of 16% per year.

a. Make a table that shows the worth of the car for years 0, 1, 2, 3, and 4. What is the real-world meaning of year 0? **the year when the car is new**

b. Which type of model best represents the data in your table? Explain. **exponential**

c. Write a function for your data. $y = 18{,}000(0.84)^x$

d. What is the value of the car after $5\frac{1}{2}$ years? **$6899.36**

e. What is the value of the car after 8 years? **$4461.77**

31. **Pet Care** The table shows general guidelines for the weight of a Great Dane at various ages.

Great Dane	
Age (mo)	Weight (kg)
2	12
4	23
6	33
8	40
10	45

31a. Possible answer: quadratic; the second differences are approximately constant at −2.

a. None of the three models in this lesson—linear, quadratic, or exponential—fits this data exactly. Which of these is the *best* model for the data? Explain your choice.

b. What would you predict for the weight of a Great Dane who is 1 year old? **about 48 kg**

c. Do you think you could use your model to find the weight of a Great Dane at any age? Why or why not?

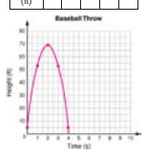
No; this quadratic model will begin to decrease although the dog's weight will either continue to grow or eventually remain constant.

PROBLEM SOLVING

Write the correct answer.

1. The table shows the height of a baseball for different times after it was thrown. Graph the data. Which kind of model best describes the data?

Height of Baseball					
Time (s)	0	1	2	3	4
Height (ft)	5	53	69	53	5



quadratic

2. The table shows the cost of peaches. Look for a pattern and determine which kind of model best describes the data. Then write a function that models the data.

Cost of Peaches				
Pounds	1	2	3	4
Cost ($)	1.29	2.58	3.87	5.16

linear

$y = 1.29x$

3. The table shows the number of computers in a school for four years.

Number of Computers				
Year	'00	'01	'02	'03
Computers	14	28	56	112

Write a function to model the data. Then use the function to predict how many computers the school will have in 2006 if the pattern continues.

$y = 14(2)^x$; 896

The chart shows the ticket sales for movies on two different screens at one theater over four days. Select the best answer.

	Screen1	Screen 2
Day 1	400	3000
Day 2	440	2400
Day 3	480	1920
Day 4	520	1536

4. Which kind of model best describes the ticket sales for the movie on screen 1?
A linear C exponential
B quadratic D none of these

5. Which function describes the data for screen 1?
F $y = 40x^2$ H $y = 400x$
G $y = 40x + 400$ J $y = 400(40)^x$

6. Which kind of model best describes the ticket sales for the movie on screen 2?
A linear C exponential
B quadratic D none of these

7. Which function describes the data for screen 2?
F $y = -600x + 3000$
G $y = 600x^2 + 2400$
H $y = 3000(0.8)^x$
J $y = 3000(1.25)^x$

CHALLENGE

If you've ridden in a car on a highway, you've probably noticed that the speed of traffic slows down when there are more cars on the road. The data at right was gathered to research this phenomenon. None of the models you have studied fits this data exactly. Furthermore, the values of the independent variable don't change by a constant amount, so you can't check for constant differences or constant ratios. You'll use a graphing calculator to find models instead.

Traffic Density (vehicles/mi)	Average Speed (mi/h)
20.4	38.8
26.5	34.0
29.5	31.8
30.0	28.8
97.0	12.3
106.2	10.6
106.7	11.4
107.2	10.3
121.7	8.5
123.9	9.8
130.5	8.6
141.3	7.7

1. Enter the data into lists **L1** and **L2** on your calculator, and make a scatter plot. (Review Technology Lab 4-5 on page 262 of your textbook if you forget how.) Sketch the scatter plot below.
2. Based on the shape of the scatter plot, which kind of model do you think will best describe the data?
exponential or quadratic
3. Press STAT and select the **CALC** menu. Select **LinReg(ax + b)** and press ENTER to find a linear model. Write the equation here, rounding to the thousandths.
$y = -0.251x + 39.469$
4. Press Y=, enter the equation for the linear model, and press GRAPH. Sketch the line on your graph above.
5. Repeat step 3, but select **QuadReg** to find a quadratic model. Write the equation here.
$y = 0.002x^2 - 0.547x + 46.625$
6. Graph the quadratic equation, and sketch it above.
7. Repeat step 3, but select **ExpReg** to find an exponential model. Write the equation here.
$y = 46.700(0.987)^x$
8. Graph the exponential equation, and sketch it above.
9. From the graphs of the three models, which model(s) seems to fit best? Explain your choice(s).
Exponential or quadratic; either model is a good fit for the curvature of the scatter plot.
10. Use each model to predict the average speed of traffic when the traffic density is 180 vehicles per mile.
linear: −5.711 mi/h quadratic: 12.965 mi/h exponential: 4.430 mi/h
11. From your answers to question 10, which model(s) seems to fit best? Explain your choice(s).
The exponential model fits best because it gives decreasing speeds as traffic density increases; the linear model gives negative speeds and the quadratic model gives increasing speeds.

TEST PREP DOCTOR In **Exercise 27,** remind students that a variable in a linear function, like the variable in choice **C,** has an exponent of one, which is usually not written.

Journal

Have students write a data set that could be modeled by an exponential function. Then have them explain how to write the function rule and how to use it to make predictions.

ALTERNATIVE ASSESSMENT

Have students create a data set that can best be modeled by a quadratic function. Have them explain why a quadratic model fits their data better than either a linear or exponential model.

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9-4 Lesson Quiz

Which kind of model best describes each set of data?

1.

Time (s)	Height of Ball (ft)
0	200
1	184
2	136
3	56

quadratic

2.

Value of Townhouse	
Age (yr)	Value ($)
0	100,000
1	102,000
2	104,040
3	106,121

exponential

3. Use the data in the table to describe how the amount of water is changing. Then write a function that models the data. Use your function to predict the amount of water in the pool after 3 hours.

Water in a Swimming Pool	
Time (min)	Amount of Water (gal)
10	327
20	342
30	357
40	372

increasing by 15 gal every 10 min; $y = 1.5x + 312$; 582 gal

Also available online

CHAPTER 9

Organizer

Pacing: Traditional 1 day
Block $\frac{1}{2}$ day

Objectives: Identify linear and nonlinear rates of change.

Compare rates of change.

Using the Extension

In tables representing functions, students have learned, given a constant *x*-difference, that a constant *y*-difference indicates a linear function and a constant *y*-ratio indicates an exponential function. In this extension, students examine whether the rate of change between ordered pairs in a function table is constant or varies, indicating a linear function or a nonlinear function.

Additional Examples

Example 1

Determine whether each function has a constant or variable rate of change.

A.

x	0	2	6	8	12
y	1	2	4	5	7

constant

B.

x	−1	0	2	3	6
y	−2	0	4	8	11

variable

Questioning Strategies

EXAMPLE 1

- In the table, which variable is the dependent variable? Which variable is the independent variable?
- Can a rate of change be positive? negative? zero?

COMMON CORE

CC.9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

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COMMON CORE

EXTENSION

Linear and Nonlinear Rates of Change

CC.9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*

Objectives
Identify linear and nonlinear rates of change.

Compare rates of change.

Recall that a *rate of change* is a ratio that compares the amount of change in a dependent variable to the amount of change in an independent variable.

$$\text{rate of change} = \frac{\text{change in dependent variable}}{\text{change in independent variable}}$$

The table shows the price of one ounce of gold in 2005 and 2008. The year is the independent variable and the price is the dependent variable. The rate of change is $\frac{870-513}{2008-2005} = \frac{357}{3} = 119$, or \$119 per year.

Price of Gold	
Year	Price ($/oz)
2005	513
2008	870

EXAMPLE 1 Identifying Constant and Variable Rates of Change

Determine whether each function has a constant or variable rate of change.

A {(0, 0), (1, 4), (3, 8), (6, 8), (8, 6)}

Find the ratio of the amount of change in the dependent variable *y* to the corresponding amount of change in the independent variable *x*.

	x	*y*	
	0	0	
+1	1	4	+4
+2	3	8	+4
+3	6	8	+0
+2	8	6	−2

The rates of change are $\frac{4}{1} = 4$, $\frac{4}{2} = 2$, $\frac{0}{3} = 0$, and $\frac{-2}{2} = -1$. The function has a variable rate of change.

B {(0, 1), (1, 2), (4, 5), (6, 7), (7, 8)}

Find the ratio of the amount of change in the dependent variable *y* to the corresponding amount of change in the independent variable *x*.

	x	*y*	
	0	1	
+1	1	2	+1
+3	4	5	+3
+2	6	7	+2
+1	7	8	+1

The rates of change are $\frac{1}{1} = 1$, $\frac{3}{3} = 1$, $\frac{2}{2} = 1$, and $\frac{1}{1} = 1$. The function has a constant rate of change.

CHECK IT OUT!

Determine whether each function has a constant or variable rate of change.

1a. {(−3, 10), (0, 7), (1, 6), (4, 3), (7, 0)} constant

1b. {(−2, −3), (2, 5), (3, 7), (5, 9), (8, 12)} variable

1 Introduce

Motivate

Remind students that they have already worked with rates of change. Ask students to give a real-world example of a rate of change. Possible answer: a tank that fills with water at the rate of 10 gallons per minute Explain that a function may have a constant rate of change or a variable rate of change.

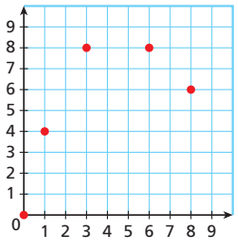
2 Teach

Guided Instruction

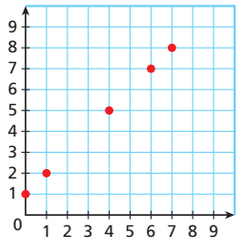
As you teach the lesson, emphasize the connection between rate of change and slope. In particular, point out that for a nonlinear function, the rate of change between two points on the graph is the slope of the line segment connecting the points. The different slopes of the different line segments illustrate why a nonlinear function has a variable rate of change.

The functions in Examples 1A and 1B are graphed below.

Example 1A (variable rate of change)



Example 1B (constant rate of change)



A function is a *linear function* if and only if the function has a constant rate of change. The graph of such a function is a straight line and the rate of change is the slope of the line, as in Example 1B.

A function with a variable rate of change, as in Example 1A, is a *nonlinear function*. Examples of nonlinear functions include quadratic functions and exponential functions.

EXAMPLE 2 Identifying Linear and Nonlinear Functions

Use rates of change to determine whether each function is linear or nonlinear.

A

x	y
−2	0
0	1
1	1.5
4	3
10	6

+2, +1, +3, +6 / +1, +0.5, +1.5, +3

Find the rates of change.

$\frac{1}{2}$ $\frac{0.5}{1} = \frac{1}{2}$ $\frac{1.5}{3} = \frac{1}{2}$ $\frac{3}{6} = \frac{1}{2}$

There is a constant rate of change, $\frac{1}{2}$, so this function is linear.

B

x	y
−6	18
−2	2
2	2
0	0
4	8

+4, +4, −2, +4 / −16, +0, −2, +8

Find the rates of change.

$\frac{-16}{4} = -4$ $\frac{0}{4} = 0$ $\frac{-2}{-2} = 1$ $\frac{8}{4} = \frac{1}{2}$

The rates of change are not constant, so this function is nonlinear.

CHECK IT OUT! **Use rates of change to determine whether each function is linear or nonlinear.**

2a. nonlinear

x	y
−2	$\frac{1}{4}$
−1	$\frac{1}{2}$
0	1
3	8
4	16

2b. linear

x	y
−5	3
−1	3
1	3
3	3
7	3

When you are given a verbal description of a function, you can determine whether the function is linear or nonlinear by making a table of values and examining the rates of change. You can compare two functions by comparing their rates of change.

COMMON ERROR ALERT

When students calculate rates of change from a table, they may calculate the ratio of the change in the independent variable to the change in the dependent variable. Students may make this error because the independent variable is usually shown above the dependent variable in a horizontal table. Remind students that the change in the dependent variable goes in the numerator.

Additional Examples

Example 2

Use rates of change to determine whether each function is linear or nonlinear.

A.

x	y
−1	1
1	1
2	−2
4	−14
5	−23

nonlinear

B.

x	y
−3	20
0	11
1	8
5	−4
10	−19

linear

MATHEMATICAL PRACTICES Questioning Strategies

EXAMPLE 2

- Can a linear function have a variable rate of change? Why or why not?

Teaching Tip **Reading Math** Be sure students realize that the word *variable* is used in two different ways in this lesson. In the terms *dependent variable* and *independent variable*, the word *variable* refers to a letter or symbol that is used to represent a value. In the term *variable rate of change*, the word *variable* means "changing."

Additional Examples

Example 3

Two auction Web sites start with 100 members each. At site A, the number of members doubles each month. At site B, 500 new members are added each month. Describe the functions that give the number of members for each site as linear or nonlinear. Which web site is growing more quickly from month 3 to month 4? site A: nonlinear; site B: linear; site A

Mathematical Practices Questioning Strategies

EXAMPLE 3

- How can you use a verbal description of a function to tell whether the function is linear or nonlinear?
- What should you compare in order to determine which function is increasing more quickly?

EXAMPLE 3 *Physical Science Application*

Two water tanks contain 512 gallons of water each. Tank A begins to drain, losing half of its volume of water every hour. Tank B begins to drain at the same time and loses 40 gallons of water every hour. Identify the function that gives the volume of water in each tank as linear or nonlinear. Which tank loses water more quickly between hour 4 and hour 5?

Use the verbal descriptions to make a table for the volume of water in each tank.

Time (h)	0	1	2	3	4	5
Water in Tank A (gal)	512	256	128	64	32	16

Time (h)	0	1	2	3	4	5
Water in Tank B (gal)	512	472	432	392	352	312

For tank A, the rates of change are –256, –128, –64, –32, and –16, so the rate of change is variable and the function is nonlinear.

For tank B, the rates of change are all –40, so the rate of change is constant and the function is linear.

Between hours 4 and 5, the volume of water in tank A decreases at a rate of 16 gallons per hour. The volume of water in tank B decreases at a rate of 40 gallons per hour. Tank B loses water more quickly.

CHECK IT OUT!

3. Reka and Charlotte each invest $500. Each month, Charlotte's investment grows by $25, while Reka's investment grows by 5% of the previous month's amount. Identify the function that gives the value of each investment as linear or nonlinear. Who is earning money more quickly between month 3 and month 4?

Reka: nonlinear; Charlotte: linear; Reka

EXTENSION Exercises

Use rates of change to determine whether each function is linear or nonlinear.

1.

x	4	5	7	10	12
y	–2	–1	1	4	6

linear

2.

x	–2	3	4	6	8
y	–4	6	8	14	20

nonlinear

3.

x	0	3	9	12	18
y	14	12	8	6	2

linear

4.

x	–8	–6	–4	–2	0
y	–3	1	3	5	9

nonlinear

3 Close

Summarize

Help students make a table like the one below to summarize the main ideas of this lesson.

Type of Function	Rate of Change	Graph
Linear	constant	line
Nonlinear	variable	not a line

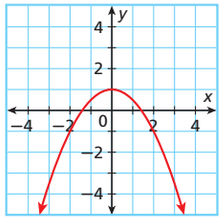
5. Caitlin: linear; Greg: nonlinear; Caitlin's collection

5. **Hobbies** Caitlin and Greg collect stamps. Each starts with a collection of 50 stamps. Caitlin adds 15 stamps to her collection each week. Greg adds 1 stamp to his collection the first week, 3 stamps the second week, 5 stamps the third week, and so on. Identify the function that gives the number of stamps in each collection as linear or nonlinear. Which collection is growing more quickly between week 5 and week 6?

Determine whether each function has a constant or variable rate of change.

6. variable
7. constant
8. variable

6.



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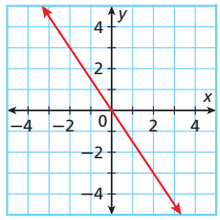

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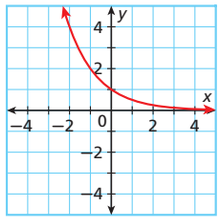


9. $y = 2x^2$ **variable**

10. $y + 1 = 3x$ **constant**

11. $y = -7$ **constant**

12. $y = \frac{1}{5}x$ **constant**

13. $y = 5^x$ **variable**

14. $y = x^2 + 1$ **variable**

15. $y = 3\sqrt{x}$ **variable**

16. $y = \frac{x-3}{2x}$ **variable**

17. $x + y = 6.25$ **constant**

Determine whether each statement is sometimes, always, or never true.

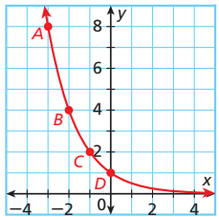
18. A function whose graph is a straight line has a variable rate of change. **never**

19. A quadratic function has a constant rate of change. **never**

20. The rate of change of a linear function is negative. **sometimes**

21. The rate of change between two points on the graph of a nonlinear function is 0. **sometimes**

22. **Critical Thinking** The figure shows the graph of the exponential function $y = \left(\frac{1}{2}\right)^x$.



22b. All of the rates of change are negative; yes.

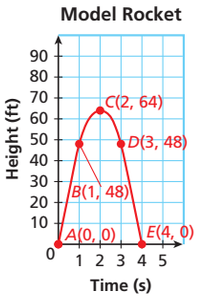
c. Since the graph moves downward from left to right, it makes sense that the rate of change between any two points is negative.

a. Find the rates of change between points A and B, between points B and C, and between points A and D. **-4; -2; $-\frac{7}{3}$**

b. What do you notice about the rates of change you found in part **a**? Do you think this would be true for the rate of change between any two points on the graph?

c. How do your findings about the rates of change relate to the shape of the graph?

23. A model rocket is launched from the ground. The graph shows the height of the rocket at various times.



23b. Between *A* and *B*; the rocket is rising more quickly between 0 s and 1 s (48 ft/s) than between 1 s and 2 s (16 ft/s).

d. The rocket is falling; the height is decreasing.

a. Find the rates of change between points A and B and between points B and C. **48; 16**

b. Which rate of change is greater? What does this tell you about the motion of the rocket?

c. Find the rates of change between points C and D and between points D and E. **-16; -48**

d. What does the sign of the rates of change you found in part **c** tell you about the motion of the rocket? Explain.

Teaching Tip

Multiple Representations If students have difficulty finding the rate of change between two points on the graph of a nonlinear function, encourage them to draw a line through the two points. This will help them visualize the slope and quickly determine whether the rate of change is positive or negative.

Extension **659**

9-5 Organizer

Pacing: Traditional 2 days
Block 1 day

Objectives: Compare functions in different representations.

Estimate and compare rates of change.

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Warm Up

Find the slope of the line that contains each pair of points.

1. (4, 8) and (−2, −10) 3

2. (−1, 5) and (6, −2) −1

Tell whether each function could be quadratic. Explain.

3. {(−1, −3), (0, 0), (1, 3), (2, 12)} yes; constant 2nd differences (6)

4. {(−2, 11), (−1, 9), (0, 7), (1, 5), (2, 3)} no; the function is linear because 1st differences are constant (−2).

Also available online

Math Humor

Q: Why couldn't the linear function and the quadratic function get along?

A: Because of their constant differences.

COMMON CORE

CC.9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).
CC.9-12.F.IF.6 Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.*
Also **CC.9-12.LE.3**

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COMMON CORE

9-5 Comparing Functions

CC.9-12.F.IF.9 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *Also* **CC.9-12.F.IF.6*, CC.9-12.F.LE.3***

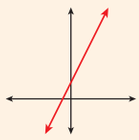
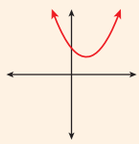
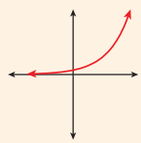
Objectives
Compare functions in different representations.

Estimate and compare rates of change.

Vocabulary
average rate of change

Who uses this?
Investment analysts can use different function representations to compare investments. (See Example 2.)

You have studied different types of functions and how they can be represented as equations, graphs, and tables. Below is a review of three types of functions and some of their key properties.

	Linear	Quadratic	Exponential
Equation	$y = mx + b$ Example: $y = 2x + 1$	$y = ax^2 + bx + c$, $a \neq 0$ Example: $y = x^2 - 2x + 3$	$y = ab^x$, $a \neq 0$, $b \neq 1$, $b > 0$ Example: $y = 0.5(2)^x$
Graph			
Table	x: 0, 1, 2, 3, 4 y: 1, 3, 5, 7, 9 +2, +2, +2, +2 Constant first differences	x: 0, 1, 2, 3, 4 y: 3, 2, 3, 6, 11 −1, +1, +3; +2, +2, +2 Constant second differences	x: 0, 1, 2, 3, 4 y: 0.5, 1, 2, 4, 8 ×2, ×2, ×2, ×2 Constant ratios

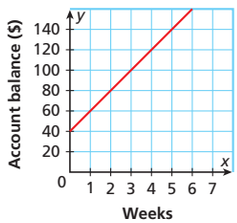
EXAMPLE 1 **Comparing Linear Functions**

Deirdre and Beth each deposit money into their checking accounts weekly. Their account information for the past several weeks is shown below.

Deirdre's Account

Weeks	Account Balance ($)
0	60
1	75
2	90
3	105
4	120

Beth's Account



1 Introduce

EXPLORATION

Comparing Functions

The monthly costs of three basic cell phone plans are described in the table below. Each plan has a monthly fee of $20 plus an additional charge for each text message. In the table, C represents the total monthly cost and m represents the number of text messages.

Number of Text Messages per Month m	Linear Plan $C = 20 + 0.10m$	Quadratic Plan $C = 20 + 0.01m^2$	Exponential Plan $C = 20(1.01)^m$
50	$25	$45	$32.89
100	$30	$120	$54.10
150	$35	$245	$88.97
200	$40	$420	$146.32
250	$45	$645	$240.64
300	$50	$920	$395.77

1. Complete the table.
2. Explain how to find the total cost C for a given number of text messages m. Substitute the number of messages for m in each equation.
3. What is the cost of each plan for 500 text messages? Linear: $70; Quadratic: $2520; Exponential: $2695.40

THINK AND DISCUSS

4. **Compare** the rate at which C increases under each type of plan for the values of m shown in the table. C increases the slowest under the Linear Plan; C increases the fastest under the Quadratic Plan.
5. **Describe** how C changes as m increases in the Linear Plan. C increases at a constant rate, of $0.10, per text message.
6. **Describe** how C changes as m increases in the Quadratic and Exponential Plans. C increases rapidly for each text message under the Quadratic Plan because the number of text messages, m, is squared. C increases more slowly under the Exponential Plan than under the Quadratic Plan for the values of m shown in the table.

Motivate

Explain to students that different types of functions can be used to show trends. Linear functions can be used to show constant changes, such as the relationship between distance, rate of speed, and time. Quadratic functions may show a rate that increases over time, peaks, and then decreases again. Each type of function shows a different behavior.

Explorations and answers are provided in the online edition.

Compare the accounts by finding slopes and y-intercepts and interpreting those values in the context of the situation.

Deirdre	Beth	Interpret and Compare.
Slope Use (1, 75) and (2, 90). $\frac{90-75}{2-1} = 15$	**Slope** Use (1, 60) and (2, 80). $\frac{80-60}{2-1} = 20$	The slope is the rate of change. Beth is saving at a higher rate.
***y*-intercept** (0, 60) y-intercept = 60	***y*-intercept** (0, 40) y-intercept = 40	The y-intercept is the beginning account balance. Deirdre started with more money.

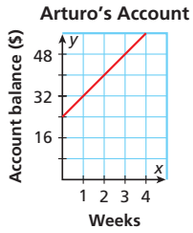
1. Slope: Dave is saving at a higher rate ($12/wk) than Arturo ($8/wk); *y*-int.: Dave started with more money ($30) than Arturo ($24).

CHECK IT OUT!

1. Dave and Arturo each deposit money into their checking accounts weekly. Their account information for the past several weeks is shown. Compare the accounts by finding and interpreting slopes and y-intercepts.

Dave's Account

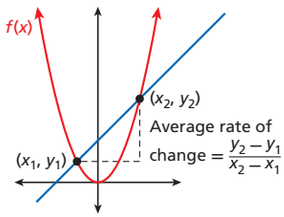
Weeks	0	1	2	3
Account Balance ($)	30	42	54	66



Reading Math

The notation [0, 20] means all x-values from 0 to 20, including 0 and 20.

Remember that nonlinear functions do not have a constant rate of change. One way to compare two nonlinear functions is to calculate their *average rates of change* over a certain interval. For a function $f(x)$ whose graph contains the points (x_1, y_1) and (x_2, y_2), the **average rate of change** over the interval $[x_1, x_2]$ is the slope of the line through (x_1, y_1) and (x_2, y_2).



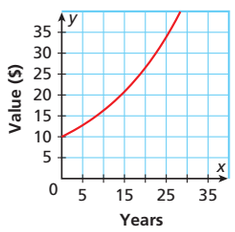
EXAMPLE 2 Comparing Exponential Functions

An investment analyst offers two different investment options for her customers. Compare the investments by finding and interpreting the average rates of change from year 0 to year 20.

Investment A

Years	Value ($)
0	10.00
5	13.38
10	17.91
15	23.97
20	32.07
25	42.92

Investment B



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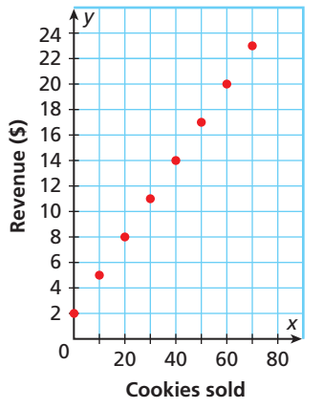
Additional Examples

Example 1

Sonia and Jackie each bake and sell cookies after school, and they each charge a delivery fee. The revenue for the sales of various numbers of cookies is shown. Compare the girls' prices by finding and interpreting the slopes and *y*-intercepts.

Sonia's Revenue	
Cookies Sold	**Revenue ($)**
24	11.00
48	17.00
72	23.00
96	29.00
120	35.00
144	41.00
168	47.00

Jackie's Revenue



Slope: Jackie charges more per cookie ($0.30) than Sonia does ($0.25); y-int.: Jackie's delivery fee ($2) is less than Sonia's ($5).

Also available online

2 Teach

Guided Instruction

Review the function types and their properties. Explain that linear and quadratic functions have constant differences, but exponential functions have constant ratios. For each type of function, discuss the average rate of change and what it means in the situation. Show that the y-intercept usually represents the initial value in a situation, such as the height of an object at 0 seconds.

Reaching All Learners

Through Visual Cues

Have students discuss the rate of change of a linear function and the average rate of change of a quadratic function. Graph a line and show them that the rate of change is the slope of the line. Then graph a quadratic or exponential function, and connect two different points on the graph with a segment. Show them that the slope of the segment is the average rate of change between the two points.

Questioning Strategies

EXAMPLE 1

- For a linear function, which quantity represents the rate of change, and which represents the starting value of x?

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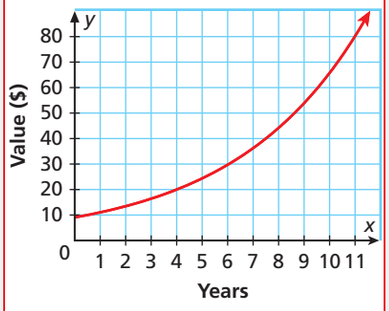
Additional Examples

Example 2

An investment analyst offers two different investment options for her customers. Compare the investments by finding and interpreting the average rates of change from year 0 to year 10.

Investment A



Investment B

Years	Value ($)
0	9.00
2	13.42
4	20.02
6	29.88
8	44.57
10	66.50
12	99.20

A increased ≈$5.60/yr; B increased $5.75/yr

Example 3

Compare the functions $y_1 = 0.35x^2 - 3x + 1$ and $y_2 = 0.3x^2 - 2x + 2$ by finding minimums, x-intercepts, and average rates of change over the x-interval [0, 10].

min.: $y_1 \approx -5.43$, $y_2 \approx -1.33$; x-int.: $y_1 \approx 0.35$, ≈ 8.22; $y_2 \approx 1.23$, ≈ 5.44; avg. rate of chg. over [0, 10]: $y_1 \approx 0.5$; $y_2 = 1$

Also available online

Calculate the average rates of change over [0, 20] by using the points whose x-coordinates are 0 and 20.

Investment A

$\frac{32.07 - 10.00}{20-0} = \frac{22.07}{20} \approx 1.10$ *Use (0, 10.00) and (20, 32.07).*

Investment B

$\frac{27 - 10}{20-0} = \frac{17}{20} = 0.85$ *Use the graph to estimate. When x = 20, y ≈ 27. Use (0, 10) and (20, 27).*

From year 0 to year 20, investment A increased at an average rate of $1.10 per year, while investment B increased at an average rate of $0.85 per year.

CHECK IT OUT! **2.** Compare the same investments' average rates of change from year 10 to year 25. **A increased about $1.67/yr; B increased about $1.13/yr.**

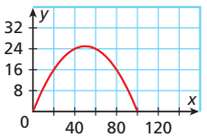
EXAMPLE 3 Comparing Quadratic Functions

Students in an engineering class were given an assignment to design a parabola-shaped bridge. Two students' models are shown at right. Compare the models by finding and interpreting maximums, x-intercepts, and average rates of change over the x-interval [0, 20].

Rosetta's Plan

$y = -0.01x^2 + 1.2x$

Marco's Plan



- The maximum is the height of the bridge at its highest point.
- The difference of the x-intercepts is the length of the bridge.
- The average rate of change over [0, 20] indicates the steepness of the bridge over the first 20 horizontal feet.

Remember! The minimum or maximum of a quadratic function is the y-value of the vertex.

For Rosetta's plan, use the function.

Step 1 Find the maximum.

$x = -\frac{b}{2a} = -\frac{1.2}{2(-0.01)} = \frac{1.2}{0.02} = 60$ *Find the x-value of the vertex.*

$y = -0.01(60)^2 + 1.2(60)$ *Substitute 60 into the function to find the y-value of the vertex.*
$= -36 + 72 = 36$

The vertex is (60, 36) and the maximum is 36.

Step 2 Find the x-intercepts.

$-0.01x^2 + 1.2x = 0$ *Write the related equation.*
$-x(0.01x - 1.2) = 0$ *Solve for x.*
$x = 0$ or $0.01x - 1.2 = 0$
$0.01x = 1.2$
$x = 120$

The x-intercepts are 0 and 120.

Step 3 Find the average rate of change over [0, 20].

At $x = 0$, $y = -0.01(0)^2 + 1.2(0) = 0$ *Find the points whose x-coordinates are 0 and 20.*

At $x = 20$, $y = -0.01(20)^2 + 1.2(20)$
$= -4 + 24 = 20$

Use (0, 0) and (20, 20): $\frac{20 - 0}{20 - 0} = 1$

MATHEMATICAL PRACTICES Questioning Strategies

EXAMPLE 2

- How can you determine the average rate of change of a given function over a certain interval?

EXAMPLE 3

- Explain why an average rate of change may be different over two different intervals.

For Marco's plan, use the graph.

Step 1 Find the maximum.

The maximum is slightly greater than 24, between 24 and 32.

Step 2 Find the x-intercepts.

The x-intercepts are 0 and 100.

Step 3 Find the average rate of change over [0, 20].

Use (0, 0) and (20, 16): $\frac{16-0}{20-0} = \frac{16}{20} = 0.8$

	Rosetta's Bridge	Marco's Bridge	Interpret and Compare.
Maximum Height	36 ft	just over 24 ft	Rosetta's bridge is taller.
Length	120 – 0 = 120 ft	100 – 0 = 100 ft	Rosetta's bridge is longer.
Average Steepness Over [0, 20]	1	0.8	Rosetta's bridge is steeper over [0, 20].

3. Rosetta's new model: max ht. = 30.25 ft; length = 110 ft; avg. steepness over [0, 20] = 0.9; new bridge is taller, longer, and steeper over [0, 20] than Marco's.

CHECK IT OUT!

3. What if...? Suppose Rosetta uses $y = -0.01x^2 + 1.1x$ and Marco uses the same plan as above. Compare Marco's model with Rosetta's new model.

EXAMPLE 4 Comparing Different Types of Functions

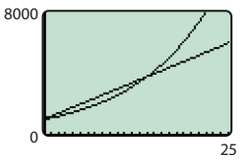
A town has approximately 1000 homes. The town council is considering plans for future development. Plan A calls for an increase of 200 homes per year. Plan B calls for a 10% increase each year. Compare the plans.

Let x be the number of years. Let y be the number of homes. Write functions to model each plan.

Plan A: $y = 200x + 1000$

Plan B: $y = 1000(1.10)^x$

Use your calculator to graph both functions.



The graphs show that under plan A, there will be more homes built than under plan B in early years.

But by the end of the 15th year, the number of homes built under plan B exceeds the number of homes built under plan A. From that point on, plan B results in more homes than plan A by ever-increasing amounts every year.

4. A: $y = 100x + 850$; B: $y = 850\ (1.08)^x$; school A's enrollment will exceed school B's enrollment at first, but school B will have more students by the end of the 11th yr. After that, school B's enrollment exceeds school A's enrollment by ever-increasing amts each yr.

CHECK IT OUT!

4. Two neighboring schools use different models for anticipated growth in enrollment: School A has 850 students and predicts an increase of 100 students per year. School B also has 850 students, but predicts an increase of 8% per year. Compare the models.

Power Presentations with PowerPoint®

Additional Examples

Example 4

A town has approximately 500 homes. The town council is considering plans for future development. Plan A calls for an increase of 50 homes per year. Plan B calls for a 5% increase each year. Compare the plans. More homes will be built under plan A up to the end of the 26th yr. After that, more homes will be built under plan B, and plan B results in more homes than plan A by ever-increasing amounts each yr.

Also available online

Questioning Strategies

EXAMPLE 4

- When comparing functions, what are the important characteristics to compare?

Teaching Tip **Reading Math** If students confuse quadratic and exponential functions, remind them that a *square* is a *quadrilateral*, and that the variable is *squared* in a *quadratic* function. In an *exponential* function, the variable is in the *exponent*.

3 Close

Summarize

Review how data in real-world situations can be compared using different functions and representations. Remind students that the rate of change of a linear function is the same for any interval. For other functions, the average rate of change will vary depending on the interval.

ONGOING ASSESSMENT and INTERVENTION

***Diagnose Before* the Lesson**
Warm Up, TE

***Monitor During* the Lesson**
Check It Out! Exercises, SE
Questioning Strategies, TE

***Assess After* the Lesson**
Lesson Quiz
Alternative Assessment

Answers to *Think and Discuss*

1. Rates of change for quadratic and exponential functions are variable, but rates of change for linear functions are constant.
2. Using an equation, you can find exact values. Using a graph, you can quickly see how variables are related.
3. See Additional Answers.

9-5 Exercises

Assignment Guide

Assign *Guided Practice* exercises as necessary.

If you finished Examples **1–2**
Basic 5, 6, 10
Average 5, 6, 9, 10
Advanced 5–6, 9, 10, 12

If you finished Examples **1–4**
Basic 5–10, 13–14, 16–30
Average 5–10, 12–14, 16–30
Advanced 5–30

Homework Quick Check
Quickly check key concepts.
Exercises: 5, 6, 7, 8, 10

Teaching Tip **Reading Math** Remind students to verify that the data provided is in the units the problem requests. For **Exercise 2**, the question asks for bacteria per hour, so students should verify that the data is given in hours.

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them. Exercises 1–10, 15

Construct viable arguments and critique the reasoning of others. Exercises 11–12

Attend to precision. Exercise 15

Learn It Online
State Resources Online

THINK AND DISCUSS

MATHEMATICAL PRACTICES

1. Explain why you need to use the word *average* when comparing rates of change for quadratic or exponential functions, but not for linear functions.
2. A function can be represented by an equation or a graph. Describe a possible advantage of each representation.
3. **GET ORGANIZED** Copy and complete the graphic organizer. Complete the sentence in each column by writing important values to compare.

Comparing Functions			
Linear to Linear	**Exponential to Exponential**	**Quadratic to Quadratic**	**Linear to Quadratic**
Compare…	Compare…	Compare…	Compare…

9-5 Exercises

GUIDED PRACTICE

SEE EXAMPLE 1

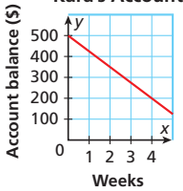
1. **Personal Finance** Fay and Kara each withdraw money from their savings accounts weekly, as shown. Compare the accounts by finding and interpreting slopes and *y*-intercepts.

Fay's Account

Weeks	0	1	2	3
Account Balance ($)	425	375	325	255

Slope: Kara is withdrawing at a higher rate ($75/wk) than Fay ($50/wk); y-int.: Kara started with more money ($500) than Fay ($425).

Kara's Account



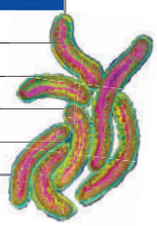
SEE EXAMPLE 2

2. **Biology** A biologist tracked the hourly growth of two different strains of bacteria in the lab. Her data are shown below. Compare the number of bacteria by finding and interpreting the average rates of change from hour 0 to hour 4.

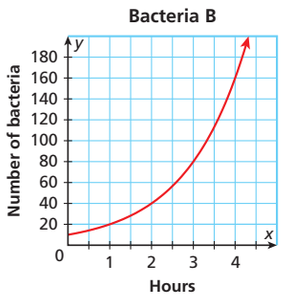
2. A: increased 100 bacteria/h; B: increased about 37.5 bacteria/h.

Bacteria A

Hours	Number of Bacteria
0	5
1	15
2	45
3	135
4	405



Bacteria B



James Cavallin/Photo Researchers, Inc

READING STRATEGIES

Functions can be compared in several ways.

Engineering students were given an assignment to design an arch. Two models are organized in a table so they can be compared. Find and interpret their maximums, *x*-intercepts, and average rates of change over the *x*-interval [0, 10].

	Model 1 $y = -0.2x^2 + 4x$	Model 2 $y = -0.02x^2 + x$	Interpret and Compare.
Maximum height of arch	$x = -\frac{b}{2a} = -\frac{4}{2(-0.2)} = \frac{4}{0.4} = 10$ $y = -0.2(10)^2 + 4(10) = 60$ The maximum height is 60 ft.	$x = -\frac{b}{2a} = -\frac{1}{2(-0.02)} = \frac{1}{0.04} = 25$ $y = -0.02(25)^2 + 25 = 12.5$ The maximum height is 12.5 ft.	The arch in model 1 is taller because the function has a greater maximum value.
Width of arch	*x*-intercepts: $-0.2x^2 + 4x = -x(0.2x - 4)$ $x = 0$ or $0.2x - 4 = 0$ $x = 0$ or 20 The width is 20 ft.	*x*-intercepts: $-0.02x^2 + x = -x(0.02x - 1)$ $x = 0$ or $0.02x - 1 = 0$ $x = 0$ or 50 The width is 50 ft.	The arch in model 2 is wider because the difference of the *x*-intercepts is greater.
Average rate of change over [0, 10]	At $x = 0$, $y = 0$. At $x = 10$, $-0.2x^2 + 4x = -0.2(10)^2 + 4(10) = 20$ Use (0, 0) and (10, 20): $\frac{20-0}{10-0} = 2$	At $x = 0$, $y = 0$. At $x = 10$, $-0.02x^2 + x = -0.02(10)^2 + 10 = 8$ Use (0, 0) and (10, 8): $\frac{8-0}{10-0} = 0.8$	The arch in model 1 is steeper than the arch in model 2 because the average rate of change is greater.

1. Look at the equation used to find the maximum height of the arch. Which variable represents the maximum height? *y*
2. Look at the equations used to find the average rate of change. Why is finding the average rate of change important? *The greater avg. rate of chg. indicates a steeper arch.*

RETEACH

Function Types

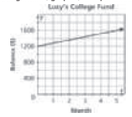
	Linear	Quadratic	Exponential
Equation	$y = mx + b$	$y = ax^2 + bx + c$ $a \neq 0$	$y = ab^x$ $a \neq 0, b \neq 1, b > 0$
Rate of change	Constant	Variable	Variable

You can compare functions in different representations, including tables, graphs, or equations.

Compare the accounts at right by finding slopes and *y*-intercepts and interpreting those values in the context of the situation.

Evie's College Fund

Month	Balance ($)
0	1000
1	1100
2	1200
3	1300
4	1400
5	1500

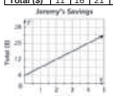


Evie	Lucy	Interpret and Compare
Slope: Use (0, 1000) and (5, 1500): $\frac{1500-1000}{5-0} = 100$	Slope: Use (0, 1200) and (5, 1600): $\frac{1600-1200}{5-0} = 80$	The slope is the rate of change. Evie is saving at a higher rate.
(0, 1000) is in the table *y*-intercept = 1000	(0, 1200) is on the graph. *y*-intercept = 1200	The *y*-intercept is the beginning account balance. Lucy started with more money.

1. Jon and Jeremy each save money weekly from their allowances, as shown. Compare the accounts by finding and interpreting slopes and *y*-intercepts.

Jon's Savings

Week	0	1	2	3	4
Total ($)	11	16	21	26	31



a. Slopes Jon: 5; Jeremy: 4
b. Interpret and compare: Jon is saving at a faster rate.
c. *y*-intercepts: Jon: $11, Jeremy: $4.
d. Interpret and compare: Jon started with more money.

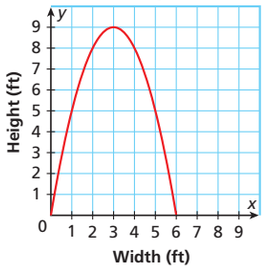
SEE EXAMPLE 3

3. **Architecture** An architect designs arch-shaped passageways in the form of a parabola. Models for two of his designs are shown at right. Compare the designs by finding and interpreting maximums, x-intercepts, and average rates of change over the interval [0, 2].

Design A

$y = -2x^2 + 8x$

Design B



SEE EXAMPLE 4

4. **Business** A bicycle store has approximately 200 bicycles in stock. The store owner is considering plans for expanding his inventory. Plan A calls for an increase of 30 bicycles per year. Plan B calls for a 10% increase each year. Compare the plans.

PRACTICE AND PROBLEM SOLVING

5. **Recreation** Kevin and Darius each hiked a mountain trail at different rates, as shown below. Compare the hikes by finding and interpreting slopes and y-intercepts.

Kevin's Hike

Time (h)	0	1	2	3
Distance from Camp (mi)	1.5	3.5	5.5	7.5

Darius's Hike

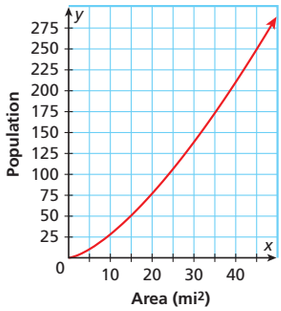
$y = 2.2x + 1$

6. **Anthropology** An archeologist used these functions to model the changing populations of two ancient cities as they grew in size. Compare the populations by finding and interpreting the average rates of change over the interval [0, 40].

City A

Area (mi²)	Population
0	0
10	31
20	89
30	164
40	252
50	353

City B



7. **Physics** Rhea's science teacher launched two model rockets straight up into the air. The functions that model the heights of the rockets are shown below. Compare the functions by finding and interpreting maximums, x-intercepts, and average rates of change over the x-interval [0, 2].

Rocket A

$y = -16x^2 + 80x$

Rocket B

Time (s)	0	1	2	3	4
Height (ft)	0	48	64	48	0

5. Slope: Darius is hiking faster (2.2 mi/h) than Kevin (2 mi/h); y-int.: Kevin started farther from camp (1.5 mi) than Darius (1 mi).

6. A: 6.3 people/mi²; B: about 5.25 people/mi²

7. A: x-int. = 0, 5; max = 100; avg. rate of chg. over [0, 2] = 48; B: x-int. = 0, 4; max = 64; avg. rate of chg. over [0, 2] = 32. A is faster, will go higher, and covers more horiz. dist. than B.

COMMON ERROR ALERT

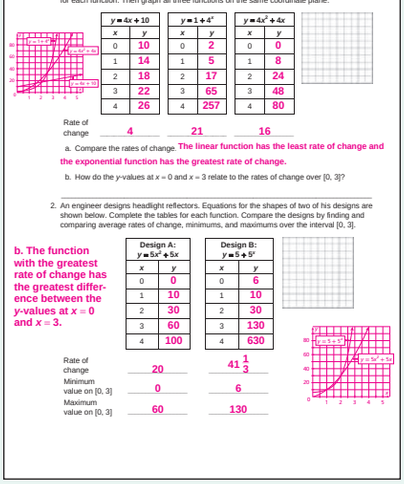
In **Exercise 7**, students may forget to find and interpret both x-intercepts. Remind them that the rocket starts on the ground, so the value at $x = 0$ is also an x-intercept.

Teaching Tip **Visual** For **Exercises 5–7**, show students each of the functions in both table and graph format. Students may be more comfortable comparing the functions if they are presented in the same format. Ask students to explain how comparing functions in the same format is different than comparing the functions given in different formats.

Answers

3. A: x-int. = 0, 4; max. = 8; avg. rate of chg. over [0, 2] = 4; B: x-int. = 0, 6; max. = 9; avg. rate of chg. over [0, 2] = 4; B is taller and wider. Both are equally steep over [0, 2].
4. Plan A will result in more bicycles at first, but plan B surpasses plan A by the end of the 8th yr. After that, B exceeds A by ever-increasing amounts each yr.

PRACTICE A
PRACTICE C
PRACTICE B


1. Three functions are given below. Complete the tables and find the rate of change over [0, 3] for each function. Then graph all three functions on the same coordinate plane.

$y = 4x + 10$

x	y
0	10
1	14
2	18
3	22
4	26

$y = 1 + 4^x$

x	y
0	2
1	5
2	17
3	65
4	257

$y = 4x^2 + 4x$

x	y
0	0
1	8
2	24
3	48
4	80

Rate of change: 4 | 21 | 16

a. Compare the rates of change. The linear function has the least rate of change and the exponential function has the greatest rate of change.

b. How do the y-values at $x = 0$ and $x = 3$ relate to the rates of change over [0, 3]?

2. An engineer designs headlight reflectors. Equations for the shapes of two of his designs are shown below. Complete the tables for each function. Compare the designs by finding and comparing average rates of change, minimums, and maximums over the interval [0, 3].

b. The function with the greatest rate of change has the greatest difference between the y-values at $x = 0$ and $x = 3$.

Design A: $y = 5x^2 + 5x$

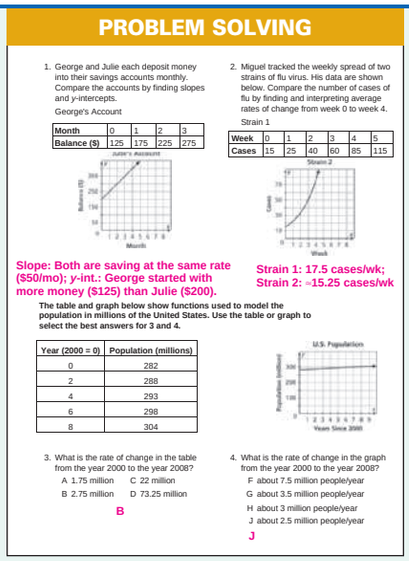
x	y
0	0
1	10
2	30
3	60
4	100

Design B: $y = 5 + 5^x$

x	y
0	6
1	10
2	30
3	130
4	630

Rate of change: 20 | 41 1/3
Minimum value on [0, 3]: 0 | 6
Maximum value on [0, 3]: 60 | 130

PROBLEM SOLVING


1. George and Julie each deposit money into their savings accounts monthly. Compare the accounts by finding slopes and y-intercepts.

George's Account

Month	0	1	2	3
Balance ($)	125	175	225	275

Slope: Both are saving at the same rate ($50/mo); y-int.: George started with more money ($125) than Julie ($200).

2. Miguel tracked the weekly spread of two strains of flu virus. His data are shown below. Compare the number of cases of flu by finding and interpreting average rates of change from week 0 to week 4.

Strain 1

Week	0	1	2	3	4	5
Cases	15	25	40	60	85	115

Strain 1: 17.5 cases/wk; Strain 2: ≈15.25 cases/wk

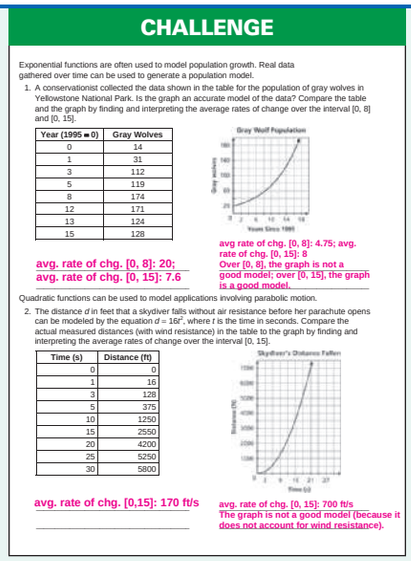
The table and graph below show functions used to model the population in millions of the United States. Use the table or graph to select the best answers for 3 and 4.

Year (2000 = 0)	Population (millions)
0	282
2	288
4	293
6	298
8	304

3. What is the rate of change in the table from the year 2000 to the year 2008?
A 1.75 million C 22 million
B 2.75 million D 73.25 million
B

4. What is the rate of change in the graph from the year 2000 to the year 2008?
F about 7.5 million people/year
G about 3.5 million people/year
H about 3 million people/year
J about 2.5 million people/year
J

CHALLENGE


Exponential functions are often used to model population growth. Real data gathered over time can be used to generate a population model.

1. A conservationist collected the data shown in the table for the population of gray wolves in Yellowstone National Park. Is the graph an accurate model of the data? Compare the table and the graph by finding and interpreting the average rates of change over the interval [0, 8] and [0, 15].

Year (1995 = 0)	Gray Wolves
0	14
1	31
3	112
5	119
8	174
12	171
13	124
15	128

avg. rate of chg. [0, 8]: 20; avg. rate of chg. [0, 15]: 7.6

avg rate of chg. [0, 8]: 4.75; avg. rate of chg. [0, 15]: 8 Over [0, 8], the graph is not a good model; over [0, 15], the graph is a good model.

Quadratic functions can be used to model applications involving parabolic motion.

2. The distance d in feet that a skydiver falls without air resistance before her parachute opens can be modeled by the equation $d = 16t^2$, where t is the time in seconds. Compare the actual measured distances (with wind resistance) in the table to the graph by finding and interpreting the average rates of change over the interval [0, 15].

Time (s)	Distance (ft)
0	0
1	16
3	128
5	375
10	1250
15	2550
20	4200
25	5250
30	5800

avg. rate of chg. [0,15]: 170 ft/s

avg. rate of chg. [0, 15]: 700 ft/s The graph is not a good model (because it does not account for wind resistance).

TEST PREP DOCTOR In **Exercise 13**, students who selected **C** found the amount of money that Tanya would save under plan A after 10 years. The result was not compared to the savings from plan B.

Journal

Have students write about the difference between a constant rate of change and an average rate of change.

ALTERNATIVE ASSESSMENT

Give students a quadratic function and ask them to represent it graphically and in table form. Then ask students to find a linear function with the same average rate of change over the interval [0, 3].

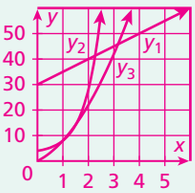
Answers

9. Possible tables:

$y_1 = 5x + 30$	
x	y
0	30
1	35
2	40
3	45
4	50
rate of chg. over [0, 4] = 5	

$y_3 = 3x^2 + 5x$	
x	y
0	0
1	8
2	22
3	42
4	68
avg. rate of chg. over [0, 4] = 17	

$y_2 = 3 + 5^x$	
x	y
0	4
1	8
2	28
3	128
4	628
avg. rate of chg. over [0, 4] = 156	



- y_1 is lin. It has the greatest y-value/ highest graph at $x = 0$. But y_1 has the least rate of chg. over [0, 4], so it has the shallowest graph and the least y-value/ lowest graph at $x = 4$.
- y_2 is exp., has the greatest rate of chg. over [0, 4], increases the most quickly, and has the steepest graph. Its graph is above the others at $x \approx 2.5$; after this pt., its graph is above the others by ever-increasing amts.
- y_3 is quad. At $x = 0$, $y_3 < y_1$ (its graph is below the graph of y_1). But y_3 has the greater rate of chg., so $y_3 > y_1$ at $x \approx 3.5$; after this pt., the graph of y_3 is steeper and higher than that of y_1 by ever-increasing amts. But y_3 does not grow as quickly as y_2. The graph of y_3 is below that of y_2 and is not as steep.

8. Proposal A will result in more boats over the first several years, but B surpasses A by the end of the 12th yr. After that, B exceeds A by ever-increasing amounts each yr.

11a. Plan B results in more students in the first few years, but plan A exceeds plan B by year 7, after which A continues to exceed B by ever-increasing amounts each year.

b. B; B will double enrollment in 6 yrs, while A will take slightly longer.

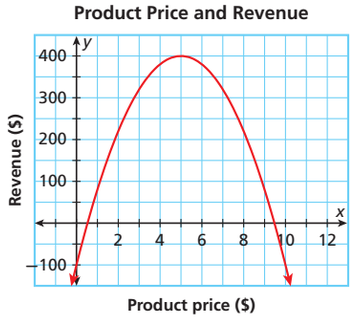
c. A; A will triple enrollment in 9–10 yrs, while B will take 12–13 yrs.

8. **Recreation** A summer boating camp has 75 boats. The camp director is considering two proposals for increasing the number of boats to match the increase in the number of campers. Proposal A recommends increasing the number of boats by 5 boats per year. Proposal B recommends a 5% increase each year. Compare the proposals.

9. Three functions are given below. Make a table of values for each function using nonnegative coordinates. Then graph all 3 functions on the same coordinate plane. Compare and interpret the tables, graphs, and rates of change over [0, 4].

$y = 5x + 30$ $y = 3 + 5^x$ $y = 3x^2 + 5x$

10. **Business** The revenue of a company based on the price of its product is modeled by the function below.



a. Estimate the average rate of change over [0, 4]. **about 120**

b. What price that will yield the maximum revenue? **$5**

11. **Critical Thinking** A karate center has 120 students. The director wants to set a goal to motivate her instructors to increase student enrollment. Under plan A, the goal is to increase the number of students by 12% each year. Under plan B, the goal is to increase the number of students by 20 each year.

a. Compare the plans.

b. Which plan should the director choose to double the enrollment in the shortest amount of time? Explain.

c. Which plan should she use to triple the enrollment in the shortest amount of time? Explain.

12. **Write About It** Compare the characteristics of linear, quadratic, and exponential functions, including any of the following that are applicable: rate of change, x-intercept, and maximum/minimum. Explain how to decide which type of function is being shown on a graph and in a table.

TEST PREP

13. Tanya has $2000 in her savings account. She wants to save more money. She is considering two savings plans. Under plan A, she will increase her account balance by $1000 per year. Under plan B, she will increase her account balance by 20% each year. How much more will she save with plan B after 10 years? Round your answer to the nearest dollar.

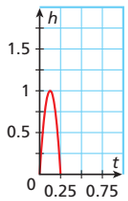
(A) $383 (C) $12,000

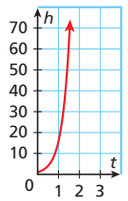
(B) $9,562 (D) $12,383

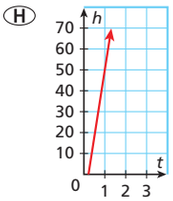
Answers

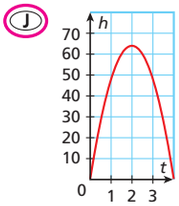
12. Linear: graph is a line; constant rate of chg.; 0 or 1 x-int.; no min. or max.; constant 1st differences. Quadratic: graph is a parabola; variable rate of chg.; 0, 1, or 2 x-int.; has min. or max. at vertex; constant 2nd differences. Exponential: variable rate of chg.; no x-int.; no min. or max.; constant ratios

14. The equation for the motion of a model rocket fired straight up with an initial velocity of 64 feet per second is $h = 64t - 16t^2$. Which could be the graph of this function?

Ⓕ 

Ⓖ 

Ⓗ 

Ⓙ 

CHALLENGE AND EXTEND

15. Two functions are shown below.

Function A

Function B

$y = 12 - 2^x$

a. Estimate the average rate of change over the intervals [0, 4], [0, 6], [4, 6], [7, 10], and [8, 12] for both functions.

b. Explain how the average rate of change varies for each function.

c. What if...? For function B, if y is doubled while x remains the same, what is the effect on the rate of change over [0, 4]?

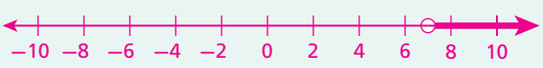
15a.

	A	B
[0, 4]	8	−3.75
[0, 6]	6	−10.5
[4, 6]	2	−24
[7, 10]	−5	−298.7
[8, 12]	−8	−960

b. A: rate of chg. is pos. and decreasing until (6, 36), when it becomes neg. and decreasing. B has ever-decreasing neg. rate of chg.

c. It is doubled.

Answers

28. 

29.

30.

31.

32.

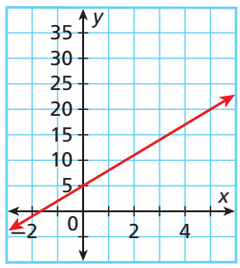
33.

Power Presentations with PowerPoint®

9-5 Lesson Quiz

1. Find the average rates of change over the interval [2, 5] for the functions shown.

Function A



A: 3; B: ≈47.01

Function B

x	y
0	1
1	2.72
2	7.39
3	20.09
4	54.60
5	148.41

2. Compare $y = x^2$ and $y = -x^2$ by finding minimums/maximums, x-intercepts, and average rates of change over the interval [0, 2]. Both have x-int. 0, which is also the max. of $y = -x^2$ and the min. of $y = x^2$. The avg. rate of chg. for $y = x^2$ is 2, which is the opp. of the avg. rate of chg. for $y = -x^2$.

3. A car manufacturer has 40 cars in stock. The manufacturer is considering two proposals. Proposal A recommends increasing the inventory by 12 cars per year. Proposal B recommends an 8% increase each year. Compare the proposals. Under proposal A, more cars will be manufactured for the first 29 yrs. After the 29th yr, more cars will be manufactured under proposal B.

Also available online

SECTION 9B

MULTI-STEP TEST PREP

Organizer

Objective: Assess students' ability to apply concepts and skills in a real-world format.

INTERACTIVE **Online Edition**

Answers

3, 5, 6. For tables and graphs, see Additional Answers.

MATHEMATICAL PRACTICES

Make sense of problems and persevere in solving them.

Learn It Online
State Resources Online

MATHEMATICAL PRACTICES **Make sense of problems and persevere in solving them.**

Exponential Functions

Dollars for Scholars In 1980, the average annual tuition at two-year colleges was $350. Since then, the cost of tuition has increased by an average of 9% each year.

1. What type of function should you use to model this situation? **exponential**
2. Write a function rule that models the annual growth in tuition at two-year colleges since 1980. Let 1980 be year zero in your function. Identify the variables, and tell which is independent and which is dependent.
3. Use your function to determine the average annual tuition in 2006. Use a table and a graph to support your answer. **$3289.71**
4. Use your function to predict the average annual tuition at two-year colleges for the year you plan to graduate from high school. **Answers will vary.**
5. In what year is the average annual tuition twice as much as in 1980? Use a table and a graph to support your answer. **about 1988**
6. In what year does the average annual tuition reach $1000? Use a table and a graph to support your answer. **1993**

(c), Laimute Druskis/Stock Boston/Aurora Photos; (bc), © D. Hurst/Alamy; (br), © Brian Hagiwara/Brand X Pictures/Alamy

2. $y = 350(1.09)^x$; y = tuition (dep.); x = years since 1980 (indep.)

INTERVENTION

Scaffolding Questions

1. Explain why it is best to use 1980 as year zero. This is the first year with given data.
2. What is the annual growth factor as a decimal? 1.09
3. How do you find your x-value? the year I graduate from high school minus 1980
4. What is the starting value for tuition? $350
5. Describe how the tuition amounts change with each additional year. Each year the cost is 9% more than the previous year.

Extension

How would the function change if the annual growth rate were 12%? The function would be $y = 350(1.12)^x$.

How would the graph change with this growth rate? It would increase more rapidly.

READY TO GO ON?

CHAPTER 9

SECTION 9B

Quiz for Lessons 9-3 Through 9-5

9-3 Exponential Growth and Decay

Write a function to model each situation. Then find the value of the function after the given amount of time.

1. Fiona's salary is $30,000, and she expects to receive a 3% raise each year; 10 years.
2. $2000 is invested at a rate of 4.5% compounded monthly; 3 years.
3. A $1200 computer is losing value at a rate of 20% per year; 4 years.
4. Strontium-90 has a half-life of 29 years. About how much strontium-90 will be left from a 100-mg sample after 290 years? Round your answer to the nearest thousandth. 0.098 mg

9-4 Linear, Quadratic, and Exponential Models

Graph each data set. Which kind of model best describes the data?

5. $\{(-2, 5), (3, 10), (0, 1), (1, 2), (0.5, 1.25)\}$ quadratic
6. $\{(0, 3), (2, 12), (-1, 1.5), (-3, 0.375), (4, 48)\}$ exponential

Look for a pattern in each data set to determine which kind of model best describes the data.

7. $\{(-2, -6), (-1, -5), (0, -4), (1, -3), (2, -2)\}$ linear
8. $\{(-2, -24), (-1, -12), (0, -6), (1, -3)\}$ exponential

9-5 Comparing Functions

9. The rocket club launched two model rockets straight up into the air. The height of Rocket A can be modeled by the function $y = -16x^2 + 88x$. The function that models the height of the Rocket B is shown in the table below. Compare the functions by finding and interpreting maximums, x-intercepts, and average rates of change over the x-interval $[0, 2]$.

Time (seconds) x	0	1	2	3	4
Height (feet) y	0	80	128	144	128

A: x-int. = 0, 5.5; max = 121; avg. rate of chg. over [0, 2] = 64; B: x-int. = 0, 6; max = 144; avg. rate of chg. over [0, 2] = 72. B is faster, will go higher, and stays in the air longer than A.

READY TO GO ON? SECTION 9B

Organizer

Objective: Assess students' mastery of concepts and skills in this section.

Resources

Assessment Resources
Section Quiz

INTERVENTION

Resources

***Ready to Go On? Intervention and Enrichment* Worksheets**

***Ready to Go On?* Online**

Answers

5, 6. For graphs, see Additional Answers.

READY TO GO ON?

Diagnose and Prescribe

NO INTERVENE

YES ENRICH

Ready to Go On? Intervention	*Ready to Go On? Intervention*, Section 9B: Worksheets	*Ready to Go On? Intervention*, Section 9B: Online	
Lesson 9-3	9-3 Intervention	Activity 9-3	Diagnose and Prescribe Online
Lesson 9-4	9-4 Intervention	Activity 9-4	
Lesson 9-5	9-5 Intervention	Activity 9-5	

***Ready to Go On? Enrichment*, Section 9B**
Worksheets
Online

Organizer

Objective: Help students organize and review key concepts and skills presented in this chapter.

Online Edition
Multilingual Glossary

Resources

Learn It Online
Multilingual Glossary

Lesson Tutorial Videos

Answers

1. geometric sequence
2. exponential decay
3. common ratio
4. exponential function
5. 81, 243, 729
6. 48, −96, 192
7. 5, 2.5, 1.25
8. −256, −1024, −4096
9. 7,812,500
10. 19,131,876

CHAPTER 9 Study Guide: Review

Vocabulary

common ratio
compound interest
exponential decay
exponential function
exponential growth
geometric sequence
half-life

Complete the sentences below with vocabulary words from the list above.

1. 3, 6, 12, 24, ... is an example of a(n) ___?___.
2. A(n) ___?___ function has the form $y = a(1 - r)^t$, where $a > 0$.
3. In the formula $a_n = a_1 r^{n-1}$, the variable r represents the ___?___.
4. $f(x) = 2^x$ is an example of a(n) ___?___.

9-1 Geometric Sequences

EXAMPLE

■ **What is the 10th term of the geometric sequence −6400, 3200, −1600, 800, ...?**

Find the common ratio by dividing consecutive terms.

$\frac{3200}{-6400} = -0.5 \quad \frac{-1600}{3200} = -0.5$

$a_n = a_1 r^{n-1}$ *Write the formula.*

$a_{10} = -6400(-0.5)^{10-1}$ *Substitute.*

$= -6400(-0.5)^9$ *Simplify.*

$= 12.5$

EXERCISES

Find the next three terms in each geometric sequence.

5. 1, 3, 9, 27, …
6. 3, −6, 12, −24, …
7. 80, 40, 20, 10, …
8. −1, −4, −16, −64, …
9. The first term of a geometric sequence is 4 and the common ratio is 5. What is the 10th term?
10. What is the 15th term of the geometric sequence 4, 12, 36, 108, ...?

9-2 Exponential Functions

EXAMPLE

■ **Tell whether the ordered pairs $\{(1, 4), (2, 16), (3, 36), (4, 64)\}$ satisfy an exponential function. Explain.**

x	y
1	4
2	16
3	36
4	64

As the x-values increase by a constant amount, the y-values are not multiplied by a constant amount. This function is not exponential.

EXERCISES

Tell whether each set of ordered pairs satisfies an exponential function. Explain.

11. $\{(0, 1), (2, 9), (4, 81), (6, 729)\}$

12. $\{(-2, -8), (-1, -4), (0, 0), (1, 4)\}$

Graph each exponential function.

13. $y = 4^x$ **14.** $y = \left(\frac{1}{4}\right)^x$

9-3 Exponential Growth and Decay

EXAMPLE

■ **The value of a piece of antique furniture has been increasing at a rate of 2% per year. In 1990, its value was $800. Write an exponential growth function to model the situation. Then find the value of the furniture in the year 2010.**

Step 1 $y = a(1 + r)^t$ *Write the formula.*

$y = 800(1 + 0.02)^t$ *Substitute.*

$y = 800(1.02)^t$ *Simplify.*

Step 2 $y = 800(1.02)^{20}$ *Substitute 20 for t.*

≈ 1188.76 *Simplify and round.*

The furniture's value will be $1188.76.

EXERCISES

15. The number of students in the book club is increasing at a rate of 15% per year. In 2001, there were 9 students in the book club. Write an exponential growth function to model the situation. Then find the number of students in the book club in the year 2008.

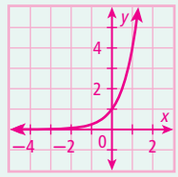
16. The population of a small town is decreasing at a rate of 4% per year. In 1970, the population was 24,500. Write an exponential decay function to model the situation. Then find the population in the year 2020.

Answers

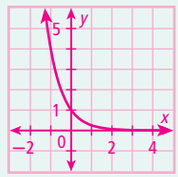
11. Yes; as the *x*-values change by a const. amt., the *y*-values are multiplied by a const. amt.

12. No; as the *x*-values change by a const. amt., the *y*-values are not multiplied by a const. amt.

13.



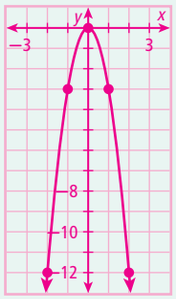
14.



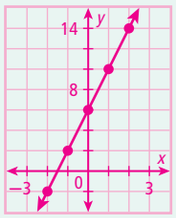
15. $y = 9(1.15)^t$; 24

16. $y = 24{,}500(0.96)^t$; 3182

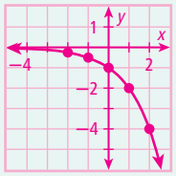
Answers

17. 

quadratic

18. 

linear

19. 

exponential

20. exponential

21. quadratic

22. linear

23. $y = 1.5x$; 15 h

9-4 Linear, Quadratic, and Exponential Models

EXAMPLE

■ **Use the data in the table to describe how Jasmin's debt is changing. Then write a function that models the data. Use your function to predict Jasmin's debt after 8 years.**

Jasmin's Debt	
Years	Debt ($)
1	130
2	260
3	520
4	1040

+1, +1, +1 (Years) × 2, × 2, × 2 (Debt)

Jasmin's debt doubles every year.

For a constant change in time (+1), there is a constant ratio of 2, so the data is exponential.

$y = ab^x$	*Write the general form.*
$y = a(2)^x$	*Substitute 2 for b.*
$130 = a(2)^1$	*Substitute (1, 130) for x and y.*
$a = 65$	*Solve for a.*
$y = 65(2)^x$	*Replace a and b in $y = ab^x$.*
$y = 65(2)^8$	*Substitute 8 for x.*
$y = 16{,}640$	*Simplify with a calculator.*

Jasmin's debt in 8 years will be $16,640.

EXERCISES

Graph each data set. Which kind of model best describes the data?

17. $\{(-2, -12), (-1, -3), (0, 0), (1, -3), (2, -12)\}$

18. $\{(-2, -2), (-1, 2), (0, 6), (1, 10), (2, 14)\}$

19. $\left\{\left(-2, -\frac{1}{4}\right), \left(-1, -\frac{1}{2}\right), (0, -1), (1, -2), (2, -4)\right\}$

Look for a pattern in each data set to determine which kind of model best describes the data.

20. $\{(0, 2), (1, 6), (2, 18), (3, 54), (4, 162)\}$

21. $\{(0, 0), (2, -20), (4, -80), (6, -180), (8, -320)\}$

22. $\{(-8, 5), (-4, 3), (0, 1), (4, -1), (8, -3)\}$

23. Write a function that models the data. Then use your function to predict how long the humidifier will produce steam with 10 quarts of water.

Input and Output of a Humidifier	
Water Volume (qt)	Steam Time (h)
3	4.5
4	6
5	7.5
6	9

9-5 Comparing Functions

EXAMPLE

Decide which linear function is increasing at a greater rate.

- Function 1 has x-intercept 4 and y-intercept–2.
- Function 2 includes the points in the table below.

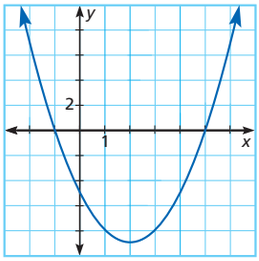
x	–2	–1	0	1
y	–11	–6	–1	4

The points (4, 0) and (0, –2) are on the graph of Function 1. Its slope of is $\frac{0-(-2)}{4-0}=\frac{1}{2}$. The table for Function 2 shows that for each increase of 1 in the value of x there is an increase of 5 in the value of y. Its slope is 5. So, Function 2 is increasing more rapidly.

EXAMPLE

Use the given information to decide which quadratic function has the lesser minimum value.

- Function 1: The function whose equation is $y = 3x^2 - 12x + 1$.
- Function 2: The function whose graph is shown.



The minimum value of Function 1 is –11. It can be found by identifying the y-value of the vertex of its parabola. The minimum value of Function 2 can be seen on the graph of the function; it is –9. So, Function 1 has the lesser minimum value.

EXERCISES

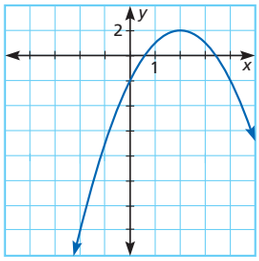
23. Decide which linear function is increasing at a greater rate.

- Function 1 has x-intercept –7 and y-intercept 4.
- Function 2 includes the points in the table.

x	0	2	4	6
y	–8	–3	2	7

24. Use the given information to decide which quadratic function has the greater maximum value.

- Function 1: The function whose equation is $y = -x^2 + 4x + 2$.
- Function 2: The function whose graph is shown.



25. Michael is studying population changes in two types of birds living on an island. Compare the populations by finding and interpreting the average rates of change over the interval [0, 18]

Bird A

Time (months)	0	6	12	18
Population (thousands)	8.3	8.6	8.8	9.1

Bird B

$y = 3.6(1.06)x$

Answers

24. The slope of Function 1 is $\frac{4}{7}$. The slope of Function 2 is $\frac{2}{5}$. Function 1 is increasing at a greater rate.

25. Function 1 has a maximum value of 6. Function 2 has a maximum value of 2. So Function 1 has the greater maximum value.

26. The population of Bird B is increasing more rapidly, but starts at a smaller population; Bird A: 44 birds per month; Bird B: 372 birds per month.

CHAPTER 9 CHAPTER TEST

Organizer

Objective: Assess students' mastery of concepts and skills in this chapter.

Online Edition

Resources

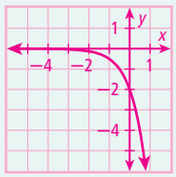
Assessment Resources

Chapte Tests

- Free Response (Levels A, B, C)
- Multiple Choice (Levels A, B, C)
- Performance Assessment

Answers

5.



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CHAPTER 9 CHAPTER TEST

Find the next three terms in each geometric sequence.

1. 2, 6, 18, 54, … 162, 486, 1458
2. 4800, 2400, 1200, 600, … 300, 150, 75
3. −4, 20, −100, 500, … −2500, 12,500, −62,500

4. **Communication** If school is cancelled, the school secretary calls 2 families. Each of those families calls 2 other families. In the third round of calls, each of the 4 families calls 2 more families. If this pattern continues, how many families are called in the seventh round of calls? 128

Graph each exponential function.

5. $y = -2(4)^x$
6. $y = 3(2)^x$
7. $y = 4\left(\frac{1}{2}\right)^x$
8. $-\left(\frac{1}{3}\right)^x$

9. A teacher is repeatedly enlarging a diagram on a photocopier. The function $f(x) = 3(1.25)^x$ represents the length of the diagram, in centimeters, after x enlargements. What is the length after 5 enlargements? Round to the nearest centimeter. 9 cm

10. Chelsea invested $5600 at a rate of 3.6% compounded quarterly. Write a compound interest function to model the situation. Then find the balance after 6 years. $A = 5600(1.009)^{4t}$; $6943.46

11. The number of trees in a forest is decreasing at a rate of 5% per year. The forest had 24,000 trees 15 years ago. Write an exponential decay function to model the situation. Then find the number of trees now. $y = 24{,}000(0.95)^t$; 11,119 trees

Look for a pattern in each data set to determine which kind of model best describes the data.

12. $\{(-10, -17), (-5, -7), (0, 3), (5, 13), (10, 23)\}$ linear
13. $\{(1, 3), (2, 9), (3, 27), (4, 81), (5, 243)\}$ exponential

14. Use the data in the table to describe how the bacteria population is changing. Then write a function that models the data. Use your function to predict the bacteria population after 10 hours. The bacteria pop. is tripling every hour; $y = 6(3)^x$; 354,294

Bacteria Population				
Time (h)	0	1	2	3
Bacteria	6	18	54	162

15. Greta is measuring how much two different candles burn over time. Compare the two candles by finding and interpreting slopes and y-intercepts.

- Candle 1 was 6 inches tall and burned for 1.5 hours.
- Candle 2 reached the heights at certain times shown in the table.

Candle 1: slope −4, y-intercept 6; Candle 2: slope −3, y-intercept 9; Candle 1 is burning faster and was shorter than Candle 2.

Time (hours)	0	0.5	1	1.5	2
Height (inches)	9	7.5	6	4.5	3

16. The information below models the population of fish living in two different lakes. Compare the populations by finding and interpreting the average rates of change over the interval [0, 50].

Fish in Lake A

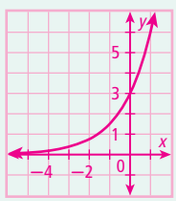
Time (years)	0	10	30	50
Population (thousands)	32	33.3	36.1	39.1

Fish in Lake B

$y = 45(1.007)^x$

The population of fish in Lake A is increasing more rapidly, and starts with a larger population; Lake A: 376 fish per month; Lake B: 142 birds per month.

6.



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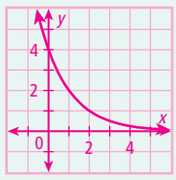

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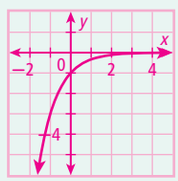


COLLEGE ENTRANCE EXAM PRACTICE

CHAPTER 9

FOCUS ON SAT MATHEMATICS SUBJECT TESTS

Colleges use standardized test scores to confirm what your academic record indicates. Because courses and instruction differ from school to school, standardized tests are one way in which colleges try to compare students fairly when making admissions decisions.

You may want to time yourself as you take this practice test. It should take you about 6 minutes to complete.

HOT TIP!

You will need to use a calculator on the SAT Mathematics Subject Tests. If you do not already have a graphing calculator, consider getting one because it may give you an advantage when solving some problems. Spend time getting used to your calculator before the test.

1. Which function models a population of 450 that is increasing by 2% per year?

(A) $y = 450(0.02)^t$

(B) $y = 450(0.2)^t$

(C) $y = 450(1.2)^t$

(D) $y = 450(1.02)^t$

(E) $y = 450(0.98)^t$

2. Which data set is best modeled by an exponential growth function?

(A) {(2, 12), (4, 23), (6, 33), (8, 40)}

(B) {(3, 0.1), (6, 0.2), (9, 0.3), (12, 0.4)}

(C) {(–1, 2), (0, 8), (1, 32), (2, 128)}

(D) {(–2, 12), (–1, 6), (0, 3), (1, 1.5)}

(E) {(0, 0), (1, 3), (2, 12), (3, 27)}

3. Brandon has $1500 in his savings account. Under plan A, he will increase his balance by $800 per year. Under plan B, he will increase his balance by 25% each year. To the nearest dollar, how much more will he save with plan B after 10 years?

(A) $383

(B) $2,476

(C) $4,470

(D) $6,392

(E) $7,162

4. The third term of a geometric sequence is 32 and the fifth term is 512. What is the eighth term of the sequence?

(A) 544

(B) 1232

(C) 8192

(D) 32,768

(E) 2,097,152

5. A band releases a new CD and tracks its sales. The table shows the number of copies sold each week (in thousands). Which type of function best models this data?

CD Sales	
Week	Copies Sold (thousands)
1	129.5
2	155
3	179.5
4	203
5	225.5
6	247

(A) Linear function

(B) Quadratic function

(C) Exponential function

(D) Square-root function

(E) Absolute-value function

Organizer

Objective: Provide practice for college entrance exams such as the SAT Mathematics Subject Tests.

INTERACTIVE **Online Edition**

Resources

College Entrance Exam Practice

Questions on the SAT Mathematics Subject Tests (Levels 1 and 2) represent the following math content areas:

	Level	
	1	2
Number and Operations	10–14%	10–14%
Algebra and Functions	38–42%	48–52%
Geometry and Measurement	38–42%	28–32%
Plane Euclidean	18–22%	0%
Coordinate	8–12%	10–14%
Three-dimensional	4–6%	4–6%
Trigonometry	6–8%	12–16%
Data Analysis, Statistics, and Probability	6–10%	6–10%

Items on this page focus on:

- Algebra and Functions

TEST PREP DOCTOR

1. Students who chose **A** used only the percent increase as a base, instead of adding that percent to 1. Students who chose **E** found the expression for a population *decreasing* by 2% per year.

2. Students who chose **B** may have confused the constant *amount* of increase of a linear function with the constant *percent* of increase of an exponential function. Students who chose **D** might have recognized it as an exponential function, but failed to realize that it is a decay function.

3. Students who chose **B** found the difference at the end of 9 years. Students who chose **D** found the difference between plan A after 12 years and plan **B** after 11 years. Students who chose **E** found the difference at the end of 11 years.

4. Students who chose **B** found the eighth term of the arithmetic sequence with the same third and fifth terms. Students who chose **E** found the ratio of a_5 to a_3 and used that as the common ratio for the sequence.

5. Students who chose **A** may have thought the first differences were close enough to being the same that a linear model would fit. However, there is a constant second difference of 1, so a quadratic model is the best fit.

CHAPTER 9 TEST TACKLER

Organizer

Objective: Provide opportunities to learn and practice common test-taking strategies.

INTERACTIVE **Online Edition**

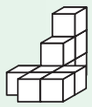
TEST PREP DOCTOR This Test Tackler focuses on how to use spatial reasoning to answer questions that require visualization of a solid figure. Before you begin, show students, using a three-dimensional model, the different views of a solid figure.

Any Question Type: Spatial Reasoning

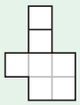
Some test questions include a three-dimensional figure. You must use spatial reasoning to correctly interpret the figure.

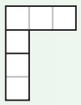
EXAMPLE 1

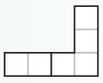
Which of the following shows the top view of the solid figure?



Ⓐ

Ⓑ 

Ⓒ 

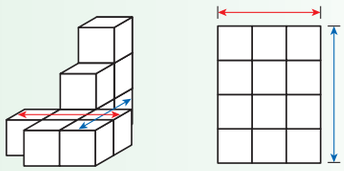
Ⓓ 

Notice that the isometric drawing shows three sides of the figure: the top, front, and right side. Use the isometric drawing to make a foundation plan of the figure.

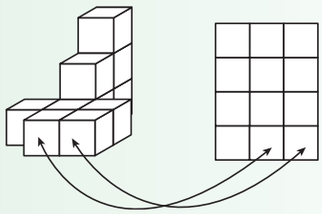
Count the number of cubes at the widest part of the figure and at the longest part of the figure.

The widest part is 3 cubes across.
The longest part is 4 cubes deep.

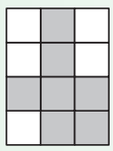
Draw a 3-by-4 grid.



Each cube on the bottom level of the figure corresponds to a square on the grid.



Start at the front of the figure. Shade in each square of the grid that corresponds to a cube on the bottom level of the figure.



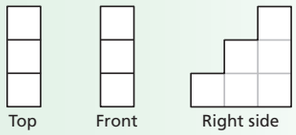
Using the foundation plan, you can see that choice B shows the top view of the solid.

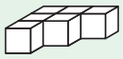
HOT TIP! Imagine that you are stacking blocks to help you to visualize the solid figure. Each square shown in the drawing represents a face of one block.

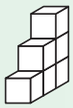
Read each test item and answer the questions that follow.

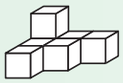
Item A

Multiple Choice Which solid is represented by these top, side, and front views?



Ⓐ 

Ⓑ 

Ⓒ 

Ⓓ

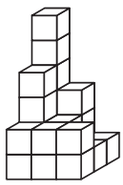
1. How many cubes can you see if you are looking down on the figure? How do you know?
2. How many cubes can you see if you are looking at the front of the figure? How do you know?
3. How many cubes high is the highest stack of the figure? Explain how you know.
4. Using your answers from Problems 1–3, what is the correct answer choice?

Item B

Short Response The drawing shows a foundation plan of a solid figure made of stacked cubes. The numbers in the squares identify the number of cubes in each stack. Draw a three-dimensional view of this solid figure.

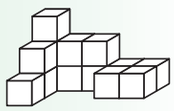
2	4	6
2	2	3
2	1	1

5. Draw the top view of the figure.
6. If you were to look at the figure from the right or from the front, how many cubes high is the highest stack? Explain how you know.
7. A student drew this figure as his response. Do you agree that this is the correct drawing? If so, explain how you know. If not, what did the student do wrong?

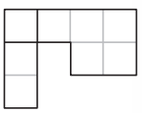


Item C

Short Response Draw the top view and front view of the solid figure.



8. If you look down on the figure, how many cubes do you see?
9. Create a base plan or foundation plan for the figure.
10. Nolan drew his version of the top view of the figure. Is he correct? If not, show how you would change his drawing so that it is correct.



11. If you look at the figure from the front, how tall is the highest stack of cubes and how short is the shortest stack of cubes?
12. Draw the front view of the figure.

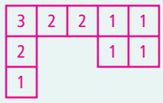
Answers to Test Items

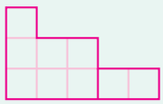
A. B

B. See answer to Problem 7.

C. See answers to Problems 10 and 12.

Answers

1. 3 cubes; the top view shows 3 squares.
2. 3 cubes; the front view shows 3 squares.
3. 3 cubes; the front view and the right side view both show a maximum of 3 cubes stacked.
4. B
5.
6. 6 cubes; the greatest number in the foundation drawing is 6.
7. No; the drawing should be rotated so that the highest stack is at the back right corner of the figure.
8. 9 cubes
9. 

10. No; I would place another square between the column with 3 squares and the column with 1 square.
11. The tallest is 3 cubes high and the shortest is 1 cube high.
12. 

Learn It Online
State Resources Online

Organizer

Objective: Provide review and practice for this chapter and standardized tests.

Online Edition

Resources

Assessment Resources
Chapter Cumulative Test

Learn It Online
State Test Practice

Learn It Online
State Resources Online

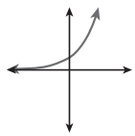
CUMULATIVE ASSESSMENT

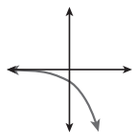
Multiple Choice

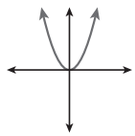
1. A sequence is defined by the rule $a_n = -3(2)^{n-1}$. What is the 5th term of the sequence?

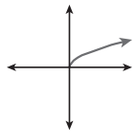
Ⓐ 5
Ⓑ −30
Ⓒ −48
Ⓓ −216

2. Which could be the graph of $y = -2^x$?

Ⓕ 

Ⓖ 

Ⓗ 

Ⓙ 

3. What is the slope of the line described by $4x - 3y = 12$?

Ⓐ 4
Ⓑ 3
Ⓒ $\frac{4}{3}$
Ⓓ $-\frac{3}{4}$

4. Which rule can be used to find any term in the sequence 8, 4, 2, 1, …?

Ⓕ $a_n = 8\left(\frac{1}{2}\right)^n$
Ⓗ $a_n = \left(\frac{1}{2}\right)^{n-1}$
Ⓖ $a_n = 2(1)^n$
Ⓙ $a_n = 8\left(\frac{1}{2}\right)^{n-1}$

5. Jerome walked on a treadmill for 45 minutes at a speed of 4.2 miles per hour. Approximately how far did Jerome walk?

Ⓐ 1.89 miles
Ⓑ 2.1 miles
Ⓒ 3.15 miles
Ⓓ 5.6 miles

6. What is the solution to the system of equations below?

$$\begin{cases} 2x - y = 2 \\ y = 3x - 5 \end{cases}$$

Ⓕ (4, 6)
Ⓗ (2, 1)
Ⓖ (3, 4)
Ⓙ (0, 2)

7. What is the complete factorization of $2x^3 + 18x$?

Ⓐ $2x(x^2 + 9)$
Ⓑ $2x(x + 3)^2$
Ⓒ $2x(x + 3)(x - 3)$
Ⓓ $2(x^3 + 18)$

8. Which ordered pair lies on the graph of $y = 3(2)^{x+1}$?

Ⓕ (−1, 0)
Ⓗ (1, 12)
Ⓖ (0, 9)
Ⓙ (3, 24)

Answers

1. C
2. G
3. C
4. J
5. C
6. G
7. A
8. H
9. B
10. F
11. 5033
12. 2
13. 272
14. 19.21
15a. exponential; $y = 3^{x-1}$
b. 19,683

When a test item gives an equation to be solved, it may be quicker to work backward from the answer choices by substituting them into the equation. If time remains, check your answer by solving the equation.

9. Simplify the expression $4(2d - 1) - 6d$.

Ⓐ $2d$

Ⓑ $2d - 4$

Ⓒ $-4d + 3$

Ⓓ $2d - 1$

10. Which is an arithmetic sequence?

I 3, 6, 9, 12, 15, …

II 1, 10, 100, 1000, …

III 4000, −2000, 1000, −500, …

Ⓕ I only

Ⓖ II and III

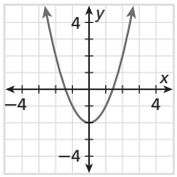
Ⓗ I and III

Ⓙ III only

Gridded Response

11. The function $f(x) = 30{,}000(0.8)^x$ gives the value of a vehicle, where x is the number of years after purchase. According to the function, what will be the value of the car in dollars 8 years after purchase? Round your answer to the nearest whole dollar.

12. The graph of $f(x)$ is shown below. How many zeros does $f(x)$ have?



13. Rosalind purchased a sewing machine at a 20%-off sale. The original selling price of the machine was $340. What was the sale price in dollars?

14. Two planes leave an airport, one heading due north and the other heading due west. After several minutes, the first plane is 12 miles north of the airport, and the second plane is 15 miles west of the airport. Estimate the distance between the two planes to the nearest hundredth of a mile.

Short Response

15. The table shows the number of people newly infected by a certain virus with one person as the original source.

Week	1	2	3	4	5	6	7
Newly Infected People	1	3	9	27	81	243	729

a. Is the data set best described by a linear function, a quadratic function, or an exponential function? Write a function to model the data.

b. Use the function to predict the number of people that will become infected in the 10th week. Show your work.

16. Ella and Mia went on a camping trip. The total cost for their trip was $124, which the girls divided evenly. Ella paid for 4 nights at the campsite and $30 for supplies. Mia paid for 2 nights at the campsite and $46 for supplies.

a. Write an equation that could be used to find the cost of one night's stay at the campsite. Explain what each variable in your equation represents.

b. Solve your equation from part **a** to find the cost of one night's stay at the campsite. Show your work.

Extended Response

17. Regina is beginning a training program to prepare for a race. In week 1, she will run 3 miles during each workout. Each week thereafter, she plans to increase the distance of her runs by 20%.

a. Write an equation to show the number of miles Regina plans to run during her workouts each week. Use the variable n to represent the week number.

b. Make a table of values to show the distances of the runs in Regina's workouts for the first 6 weeks. Round each distance to the nearest hundredth of a mile. Then graph the points.

c. Explain how to use your equation from part **a** to determine how many miles Regina will run during each workout in week 8. Then find this number.

d. Explain how to the use your graph from part **b** to determine how many miles Regina will run during each workout in week 8. How does the graph verify the answer you found in part **c**?

Short-Response Rubric

Items 15–16

2 Points = The student's answer is an accurate and complete execution of the task or tasks.

1 Point = The student's answer contains attributes of an appropriate response but is flawed.

0 Points = The student's answer contains no attributes of an appropriate response.

Extended-Response Rubric

Item 17

4 Points = The student writes the correct equation in part **a,** creates the correct table of values and graphs the points correctly in part **b,** explains how to use the equation and finds correct answer in part **c,** and explains how to use the graph to verify the algebraic answer in part **d.**

3 Points = The student writes the correct equation in part **a,** creates the correct table of values and graphs the points in part **b,** finds the correct answer in part **c** but does not explain how to use the equation, and somewhat explains how to use the graph to verify the algebraic answer in part **d.**

2 Points = The student writes the equation with a minor error in part **a,** creates a table of values and graphs the points (wrong, but correct for their equation) in part **b,** explains how to use the equation and finds an answer in part **c,** and explains how to use the graph to verify their answer in part **d.**

1 Point = The student writes a completely incorrect equation in part **a** and attempts to answer all parts, but struggles because part of their equation doesn't make sense.

0 Points = The student does not attempt all parts of the problem.

Answers

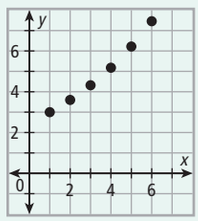
16a. $4n + 30 = 2n + 46$; n represents the cost of one night's stay at the campsite.

b. $8.00

17a. $y = 3(1.2)^{n-1}$

b.

Week	1	2	3	4	5	6
Length of Run (mi)	3.00	3.60	4.32	5.18	6.22	7.46



c. Use $n = 8$ in the equation from part **a;** 10.75 mi

d. Draw a smooth curve through the points. Use the curve to estimate the y-value when $n = 8$. This y-value is between 10 and 11, which verifies the answer to part **c**.